

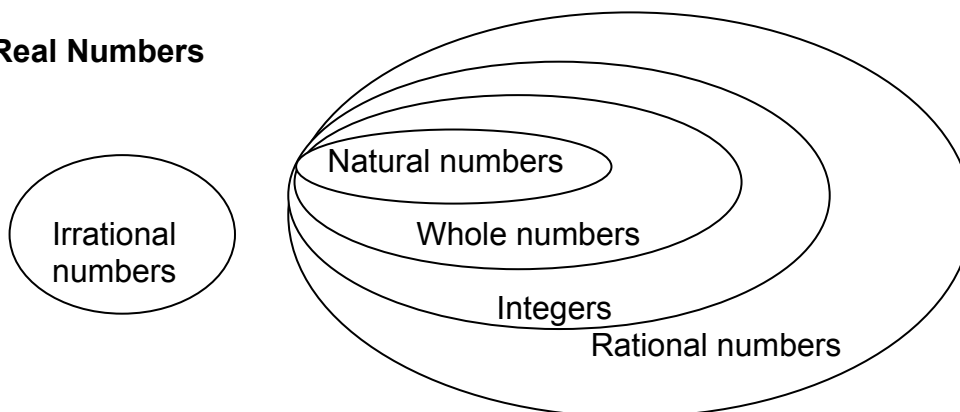
**ALGEBRA (SMR Domain 1)**

**Algebraic Structures (SMR 1.1)**

**Skill a.** Apply basic properties of real and complex numbers in constructing mathematical arguments (e.g., if  $a < b$  and  $c < 0$ , then  $ac > bc$ )

**Basic Properties of Real and Imaginary/Complex Numbers**

**Real Numbers**



For a listing of mathematical symbols, go to:  
[http://en.wikipedia.org/wiki/Math\\_symbols](http://en.wikipedia.org/wiki/Math_symbols)

**Real numbers are denoted by  $\mathbb{R}$**  and are numbers that can be shown by an infinite decimal representation such as 3.286275347.... The real numbers include rational numbers such as 42 and  $-23/129$ , and irrational numbers, such as the  $\sqrt{2}$  and  $\pi$ , which can be represented as points along an infinite number line. Real numbers are also known as, “the unique complete Archimedean *ordered field*.” Real numbers are differentiated from imaginary numbers.

**Real numbers and their properties:**

Real numbers have the two primary properties of being an ordered field, and having the least upper bound property. The First Property says that real numbers comprise an *ordered field* (with addition and multiplication as well as division by nonzero numbers), that can be ordered on a number line in a way that works with addition and multiplication.

Example:  $O$  is an **ordered field** if the order satisfies the following properties:

- if  $a \leq b$  then  $a + c \leq b + c$
- if  $0 \leq a$  and  $0 \leq b$  then  $0 \leq a b$

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It then follows that for every  $a, b, c, d$  in  $O$ :

- Either  $-a \leq 0 \leq a$  or  $a \leq 0 \leq -a$ .
- We can add inequalities:

Example: If  $a \leq b$  and  $c \leq d$ , then  $a + c \leq b + d$

- We are allowed to "multiply inequalities with positive elements":  
If  $a \leq b$  and  $0 \leq c$ , then  $ac \leq bc$ .

The Second Property of real numbers says that if a nonempty set of real numbers has an upper bound (e.g.  $\leq$  or less than or  $=$  to) then it has a least upper bound. These two properties together define the real numbers completely, and allow its other properties to be inferred: every polynomial of odd degree with real coefficients has a real root. If you add the square root of  $-1$  to the real numbers, you have a complex number and the result is algebraically closed.

Real numbers are classified as follows:

**A. Natural numbers, denoted by  $\mathbb{N}$ :** The counting numbers, 1, 2, 3, ...

Algebraic properties of natural numbers (also discussed in section 0005, Natural numbers):

	Addition	multiplication
<u>closure:</u>	$a + b$ is a natural number	$a \times b$ is a natural number
<u>associativity:</u>	$a + (b + c) = (a + b) + c$	$a \times (b \times c) = (a \times b) \times c$
<u>commutativity:</u>	$a + b = b + a$	$a \times b = b \times a$
existence of an <u>identity element:</u>	$a + 0 = a$	$a \times 1 = a$
<u>distributivity:</u>		$a \times (b + c) = (a \times b) + (a \times c)$
No <u>zero divisors:</u>		if $ab = 0$ , then either $a = 0$ or $b = 0$ (or both)

**B. Whole numbers:** The counting numbers along with zero, 0, 1, 2, ...

**C. Integers denoted by  $\mathbb{Z}$ :** The counting numbers, their negatives, and zero, ...,  $-1, 0, 1, \dots$

**D. Rationals denoted by  $\mathbb{Q}$ :** All of the fractions that can be formed from the whole numbers. Zero cannot be the denominator. In decimal form, these numbers will either be terminating or repeating decimals. Simplify square roots to determine if the number can be written as a fraction.

**E. Irrationals denoted by  $\mathbb{I}$ :** Real numbers that cannot be written as a fraction. The decimal forms of these numbers are neither terminating nor repeating. Examples:  $\pi, e, \sqrt{2}$ , etc.

**The relative size of real numbers expressed as fractions, decimals, percents and scientific notation:**

Compare the relative size of real numbers expressed in a variety of forms, including fractions, decimals, percents, and scientific notation:

To convert a fraction to a decimal, simply divide the numerator (top) by the denominator (bottom). Use long division if necessary.

If a decimal has a fixed number of digits, the decimal is said to be terminating. To write such a decimal as a fraction, first determine what place value the farthest right digit is in, for example: tenths, hundredths, thousandths, ten thousandths, hundred thousands, etc. Then drop the decimal and place the string of digits over the number given by the place value.

If a decimal continues forever by repeating a string of digits, the decimal is said to be repeating. To write a repeating decimal as a fraction, follow these steps:

1. Let  $x =$  the repeating decimal  
(e.g.  $x=0.716716716\dots$ )
2. Multiply  $x$  by the multiple of ten that will move the decimal just to the right of the repeating block of digits.  
(e.g.  $1000x=716.716716\dots$ )
3. Subtract the first equation from the second.  
(e.g.  $1000x-x=716.716716\dots-0.716716\dots$ )
4. Simplify and solve this equation. The repeating block of digits will subtract out.  
(e.g.  $999x = 716$  so  $x = \frac{716}{999}$ )

The solution will be the fraction for the repeating decimal.

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A **decimal** can be converted to a **percent** by multiplying by 100%, or merely moving the decimal point two places to the right. A **percent** can be converted to a **decimal** by dividing by 100%, or moving the decimal point two places to the left.

Examples: Convert the following decimals into percents.

$$0.375 = 37.5\%$$

$$0.7 = 70\%$$

$$0.04 = 4\%$$

$$3.15 = 315\%$$

Examples: Convert the following percents into decimals.

$$84\% = 0.84$$

$$3\% = 0.03$$

$$60\% = 0.6$$

$$110\% = 1.1$$

$$\frac{1}{2}\% = 0.5\% = 0.005$$

A **percent** can be converted to a **fraction** by placing it over 100 and reducing to simplest terms.

Examples: Convert the following percents into fractions.

$$32\% = \frac{32}{100} = \frac{8}{25}$$

$$6\% = \frac{6}{100} = \frac{3}{50}$$

$$111\% = \frac{111}{100} = 1\frac{11}{100}$$

$$= \frac{10}{100} = \frac{1}{10}$$

To find the **decimal** equivalent of a **fraction**, use the denominator to divide the numerator. Note decimal comes from deci or part of ten.

Example: Find the decimal equivalent of  $\frac{7}{10}$ .

$$\begin{array}{r} 0.7 \\ 10 \overline{)7.0} \\ \underline{70} \\ 00 \end{array}$$

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Since 10 cannot divide into 7 evenly, put a decimal point in the answer row on top; put a zero behind 7 to make it 70. Continue the division process. If a remainder occurs, put a zero by the last digit of the remainder and continue the division.

$$\text{Thus } \frac{7}{10} = 0.7$$

It is a good idea to write a zero before the decimal point so that the decimal point is emphasized.

Example: Find the decimal equivalent of  $\frac{7}{125}$ .

$$\begin{array}{r} 0.056 \\ 125 \overline{)7.000} \\ \underline{625} \phantom{00} \\ 750 \phantom{0} \\ \underline{750} \\ 0 \end{array}$$

A **decimal** can be converted into a **fraction** by multiplying by 1 in the form of a fraction (e.g.  $\frac{10}{10}$ ,  $\frac{100}{100}$ ,  $\frac{1000}{1000}$ ) to get rid of the decimal point.

Example: Convert 0.056 to a fraction.

Multiplying 0.056 by  $\frac{1000}{1000}$  to get rid of the decimal point

$$0.056 \times \frac{1000}{1000} = \frac{56}{1000} = \frac{7}{125}$$

The **percentage** of a number can be found by converting the percentage into decimal form and then multiplying the decimal by the number.

Example: Find 23% of 1000.

$$23\% = 0.23$$

$$0.23 \times 1000 = 230$$

$$= 0.0625 = 0.0625 \times \frac{10000}{10000} = \frac{625}{10000} = \frac{1}{16}$$

**Scientific notation** is a more convenient method for writing very large and very small numbers. It employs two factors. The first factor is a number between -10 and 10. The second factor is a power of 10. This notation is a shorthand way to express large numbers (like the weight of 100 freight cars in kilograms) or small numbers (like the weight of an atom in grams).

Recall:

$10^n = (10)^n$  Ten multiplied by itself  $n$  times.

$10^6 = 1,000,000$  (mega)

$10^3 = 10 \times 10 \times 10 = 1000$  (kilo)

$10^2 = 10 \times 10 = 100$  (hecto)

$10^1 = 10$  (deca)

$10^0 = 1$  Any nonzero number raised to power of zero is 1.

$10^{-1} = 1/10$  (deci)

$10^{-2} = 1/100$  (centi)

$10^{-3} = 1/1000$  (milli)

$10^{-6} = 1/1,000,000$  (micro)

**Scientific notation format.** Convert a number to a form of  $b \times 10^n$  where  $-10 < b < 10$  and  $n$  is an integer.

Example: 356.73 can be written in various forms.

$$356.73 = 3567.3 \times 10^{-1} \quad (1)$$

$$= 35673 \times 10^{-2} \quad (2)$$

$$= 35.673 \times 10^1 \quad (3)$$

$$= 3.5673 \times 10^2 \quad (4)$$

$$= 0.35673 \times 10^3 \quad (5)$$

Only (4) is written in proper scientific notation format.

Example: Write 46,368,000 in scientific notation.

1) Introduce a decimal point.  $46,368,000 = 46,368,000.0$

2) Move the decimal place to **left** until only one nonzero digit is in front of it, in this case between the 4 and 6.

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- 3) Count the number of digits the decimal point moved, in this case 7. This is the  $n^{\text{th}}$  the power of ten and is **positive** because the decimal point moved **left**.

$$\text{Therefore, } 46,368,000 = 4.6368 \times 10^7$$

Example: Write 0.00397 in scientific notation.

- 1) Decimal point is already in place.
- 2) Move the decimal point to the **right** until there is only one nonzero digit in front of it, in this case between the 3 and 9.
- 3) Count the number of digits the decimal point moved, in this case 3. This is the  $n^{\text{th}}$  the power of ten and is **negative** because the decimal point moved **right**.

$$\text{Therefore, } 0.00397 = 3.97 \times 10^{-3}.$$

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Example: Evaluate  $\frac{3.22 \times 10^{-3} \times 736}{0.00736 \times 32.2 \times 10^{-6}}$

Since we have a mixture of large and small numbers, convert each number to scientific notation:

$$736 = 7.36 \times 10^2$$

$$0.00736 = 7.36 \times 10^{-3}$$

$$32.2 \times 10^{-6} = 3.22 \times 10^{-5} \quad \text{thus, we have,}$$

$$\begin{aligned} & \frac{3.22 \times 10^{-3} \times 7.36 \times 10^2}{7.36 \times 10^{-3} \times 3.22 \times 10^{-5}} \\ &= \frac{3.22 \times 7.36 \times 10^{-3} \times 10^2}{7.36 \times 3.22 \times 10^{-3} \times 10^{-5}} \\ &= \frac{3.22 \times 7.36}{7.36 \times 3.22} \times \frac{10^{-1}}{10^{-8}} \\ &= \frac{3.22 \times 7.36}{3.22 \times 7.36} \times 10^{-1} \times 10^8 \\ &= \frac{23.6992}{23.6992} \times 10^7 \\ &= 1 \times 10^7 = 10,000,000 \end{aligned}$$

**Algebraic Structures (SMR 1.1)**

**Skill b.** Know that the rational numbers and real numbers can be ordered and that the complex numbers cannot be ordered, but that any polynomial equation with real coefficients can be solved in the complex field

**Imaginary and complex numbers and their properties:** We just reviewed real numbers. Real numbers can be ordered but complex

numbers cannot be ordered. **Complex numbers denoted by  $\mathbb{C}$ :**  $\mathbb{C}$  means  $\{a + bi : a, b \in \mathbb{R}\}$  ( $\in$  means “element of”). In other words, complex numbers are an extension of real numbers made by attaching an imaginary number  $i$ , which satisfies:

$i^2 = -1$ . Complex numbers are of the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ ; “ $a$ ” and “ $b$ ” are the real part of the complex number while “ $i$ ” is the imaginary part. When  $i$  appears in a fraction, the fraction is usually simplified so that  $i$  is not in the denominator.

The complex plane and complex numbers as ordered pairs: Every complex number  $a + bi$  can be shown as a pair of 2 real numbers. For the real part  $a$  and the imaginary part  $bi$ ,  $b$  is also real.

Example:  $3i$  has a real part 0 and imaginary part 3 and  $4$  has a real part 4 and an imaginary part 0. As another way of writing complex numbers, we can write them as ordered pairs:

Complex #	Ordered pair
$3 + 2i$	$(3, 2)$
$\sqrt{3} + \sqrt{3}i$	$(\sqrt{3} + \sqrt{3})$
$7i$	$(0, 7)$
$\frac{6 + 2i}{7}$	$\left(\frac{6}{7}, \frac{2}{7}\right)$

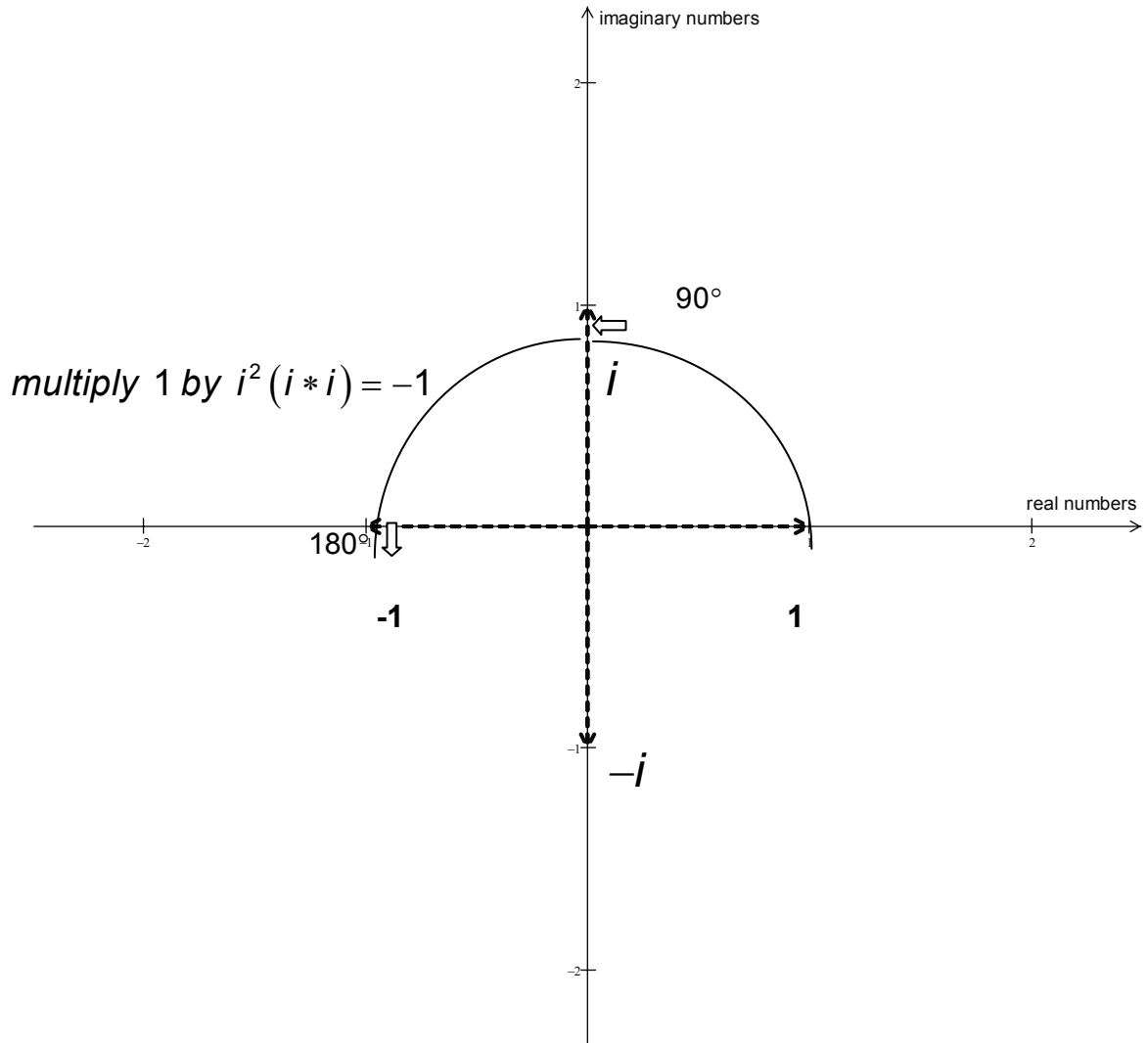
**A visual representation of imaginary numbers:**

When  $i^2$  appears in a problem, it can be replaced by -1 since

$$i^2 = -1, \quad x^2 = -1$$

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How do we turn a 1 into a -1? We can work with what is called a complex plane and visually see how it can be done. Instead of an x and a y-axis, we have an axis that represents the dimension of real numbers and an axis that represents the imaginary dimension. If we rotate  $x 180^\circ$  in a counterclockwise direction, we can change 1 into -1, which is the same as multiplying 1 by  $i$ .



We could also rotate clockwise  $180^\circ$  to turn -1 into 1. This is a multiplication by  $i^2$ .

If we multiply by  $-i$  once, we turn 1 into  $-i$  and  $-i$  into -1. Therefore, there are *two* square roots of -1:  $i$  and  $-i$ .

*Therefore,  $i$  (or  $-i$ ) is what real numbers turn into when rotated  $180^\circ$  and two rotations in either direction is -1: it brings us back to the dimension of real positive and negative numbers.*

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### Operations involving polynomial equations in a complex field:

To add or subtract complex numbers, add or subtract the real parts. Then add or subtract the imaginary parts and keep the  $i$  (just like combining like terms).

Examples: Add  $(2 + 3i) + (-7 - 4i)$ .

$$2 + -7 = -5 \qquad 3i + -4i = -i \text{ so,}$$

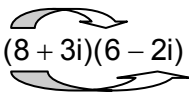
$$(2 + 3i) + (-7 - 4i) = -5 - i$$

Subtract  $(8 - 5i) - (-3 + 7i)$

$$8 - 5i + 3 - 7i = 11 - 12i$$

To multiply 2 complex numbers, F.O.I.L. (F=first terms, O= outer terms, I= inner terms and L = last terms) the 2 complex factors together. Replace  $i^2$  with  $-1$  and finish combining like terms. Answers should have the form  $a + bi$ .

Multiply the 1<sup>st</sup> terms to get 48

Example: Multiply  $(8 + 3i)(6 - 2i)$  use F.O.I.L. (First, Outer, Inner, Last). 

Multiply the outer terms to get  $-16i$  and so on....

$$48 - 16i + 18i - 6i^2 \qquad \text{Let } i^2 = -1.$$

$$48 - 16i + 18i - 6(-1)$$

$$48 - 16i + 18i + 6$$

$$54 + 2i$$

This is the answer.

Example: Multiply  $(5 + 8i)^2$  ← Write this out twice.

$$(5 + 8i)(5 + 8i) \qquad \text{F.O.I.L. this}$$

$$25 + 40i + 40i + 64i^2 \qquad \text{Let } i^2 = -1.$$

$$25 + 40i + 40i + 64(-1)$$

$$25 + 40i + 40i - 64$$

$$-39 + 80i$$

This is the answer.

When dividing 2 complex numbers, you must eliminate the complex number in the denominator. If the complex number in the denominator is of the form  $b + ci$ , multiply both the numerator and denominator by  $a - bi$ . Remember to replace  $i^2$  with  $-1$  and then continue simplifying the fraction.

Example:

$$\frac{2+3i}{5i} \quad \text{Multiply this by } \frac{i}{i}$$

$$\frac{2+3i}{5i} \times \frac{i}{i} = \frac{(2+3i)i}{5i \cdot i} = \frac{2i+3i^2}{5i^2} = \frac{2i+3(-1)}{-5} = \frac{-3+2i}{-5} = \frac{3-2i}{5}$$

If the complex number in the denominator is of the form  $a + bi$ , multiply both the numerator and denominator by **the conjugate of the denominator** (see more about the conjugate in SMR1.2, section b). The conjugate of a complex number is the number with the sign of its imaginary part reversed. The conjugate of  $2 - 3i$  is  $2 + 3i$ . The conjugate of  $-6 + 11i$  is  $-6 - 11i$ . Multiply together the factors on the top and bottom of the fraction. Remember to replace  $i^2$  with a  $-1$ , combine like terms, and then continue simplifying the fraction.

Example:

$$\frac{4+7i}{6-5i} \quad \text{Multiply numerator and denominator by } 6+5i, \\ \text{the conjugate of the denominator.}$$

$$\frac{(4+7i)(6+5i)}{(6-5i)(6+5i)} = \frac{24+20i+42i+35i^2}{36+30i-30i-25i^2} = \frac{24+62i+35(-1)}{36-25(-1)} = \frac{-11+62i}{61}$$

Example:

$$\frac{24}{-3-5i} \quad \text{Multiply numerator and denominator by } -3+5i, \\ \text{the conjugate of the denominator.}$$

$$\frac{24}{-3-5i} \times \frac{-3+5i}{-3+5i} = \frac{-72+120i}{9-25i^2} = \frac{-72+120i}{9+25} = \frac{-72+120i}{34} = \frac{-36+60i}{17}$$

**Solving polynomial equations in the complex field:**

$$4x^2 - 2x + 7 = 0$$

This is not a perfect square

$$x^2 - \frac{1}{2}x + \frac{7}{4} = 0$$

Divide by 4, the coefficient of the quadratic term

$$x^2 - \frac{1}{2}x = -\frac{7}{4}$$

Put in the form  $x^2 + bx$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = -\frac{7}{4} + \frac{1}{16}$$

To complete the square, divide  $b(2)$  by 2 and

$$\text{then square the result: } \left(-\frac{1}{2} \div 2\right)^2 = \frac{1}{16}$$

$$\left(x^2 - \frac{1}{4}\right)^2 = -\frac{27}{16}$$

Factor the left side so you have a perfect square,  
simplify the right

$$\left(x - \frac{1}{4}\right) = \sqrt{-\frac{27}{16}}$$

Use the Square Root Property

$$x - \frac{1}{4} = \pm \frac{3i\sqrt{3}}{4}$$

Simplify the right side,  $\sqrt{-1} = i$

$$x = \frac{1}{4} \pm \frac{3i\sqrt{3}}{4}$$

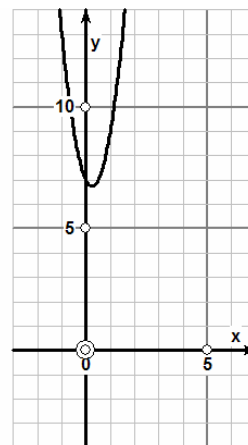
Add  $\frac{1}{4}$  to each side to solve for  $x$

The solution is  $x = \left\{ \frac{1}{4} + \frac{3i\sqrt{3}}{4}, \frac{1}{4} - \frac{3i\sqrt{3}}{4} \right\}$  an imaginary solution with

complex numbers.

Check the solution by graphing and notice that there are no real solutions for the equation as it has no roots ( $x$ -intercepts). Imaginary solutions must be checked by substituting the answers back into the original equation.

**For more on solving polynomials in the complex field, see Skill SMR1.2.b**



*quadratic equation with no real roots*

**Algebraic Structures (SMR 1.1)**

**Skill c. Know why the real and complex numbers are each a field, and that particular rings are not fields (e.g., integers, polynomial rings, matrix rings)**

**Fields and Rings:**

**A. Fields:** A set of elements (usually numbers) that may be combined under the operations of addition and multiplication so that it constitutes an additive group, the nonzero elements form a multiplicative group, and multiplication distributes over addition. The set of real numbers and the set of complex numbers are both examples of fields. A field is a ring where multiplication is commutative (in the U.S. and Russia, but not in France and other places in Europe), commutative meaning if  $a(b) = c$  then,  $b(a) = c$ .

$\mathbb{Z}$  (integers) is a ring that is not a field in that it does not have the multiplicative inverse:

For all  $a$  in  $\mathbb{F}$ ,  $a(a^{-1}) = a^{-1}(a) = 1$ , this is what multiplicative inverse means. For integers, which includes the negative numbers, the multiplicative inverse does not apply:

$$0(0^{-1}) \neq 1.$$

**B. Ring:** An algebraic structure in which addition and multiplication are defined and have properties listed below. A ring is a generalization of the set of integers, which are one example of a ring. If a ring also has the commutative property of multiplication (with zero removed) it is a field. Other examples include the polynomials and matrices.

A ring is a set  $R$  equipped with two binary operations  $+$  and  $\cdot$ , called *addition* and *multiplication*:

Multiplication distributes over addition:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c) \quad \text{Distributive Property of Multiplication}$$

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$(R, +)$

$(a + b) + c = a + (b + c)$       Associative Property of Addition

$a + b = b + a$       Commutative Property of Addition

$0 + a = a + 0 = a$       Additive Identity

$\forall a \exists (-a)$  such that  $a + -a = -a + a = 0$

$(R, *)$

$1 \cdot a = a \cdot 1 = a$       Multiplicative Identity

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$       Associative Property of Multiplication

$0a = a0 = 0$

$(-1)a = -a$

$(-a)b = a(-b) = -(ab)$

$(ab)^{-1} = b^{-1} a^{-1}$  if both  $a$  and  $b$  are invertible

**Summary of Field Axioms:**

Name	Addition	Multiplication
Commutativity	$a + b = b + a$	$ab = ba$
Associativity	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Distributivity	$a(b + c) = ab + bc$	$(a + b)c = ac + bc$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = a = 1 \cdot a$
Inverses	$a + (-a) = 0 = (-a) + a$	$a(a^{-1}) = 1 = a^{-1} a$ if $a \neq 0$

**Polynomial Equations and Inequalities (SMR 1.2)**

**Skill a. Know why graphs of linear inequalities are half planes and be able to apply this fact (e.g. linear programming)**

**Linear Inequalities:**

To graph **an inequality**, solve the inequality for  $y$ . This gets the inequality in the **slope intercept form**, (for example:  $y < m x + b$ ). The point  $(0,b)$  is the  $y$ -intercept and  $m$  is the line's slope.

- If the inequality solves to  $x > , \geq , < \text{ or } \leq$  **any number**, then the graph includes a **vertical line**.

- If the inequality solves to  $y > , \geq , < \text{ or } \leq$  **any number**, then the graph includes a **horizontal line**.

- When graphing a linear inequality, the line will be dotted if the inequality sign is  $<$  or  $>$ . If the inequality sign is either  $\geq$  or  $\leq$ , the line on the graph will be a solid line. Shade above the line when the inequality sign is  $\geq$  or  $>$ . Shade below the line when the inequality sign is  $<$  or  $\leq$ . For inequalities of the forms  $x >$  number,  $x \leq$  number,  $x <$  number, or  $x \geq$  number, draw a vertical line (solid or dotted). Shade to the right for  $>$  or  $\geq$ . Shade to the left for  $<$  or  $\leq$ .

**Remember: Dividing or multiplying by a negative number will reverse the direction of the inequality sign.**

Use these rules to graph and shade each inequality. The solution to a system of linear inequalities consists of the part of the graph where the shaded areas for all the inequalities in the system overlap. For instance, if the graph of one inequality was shaded with red, and the graph of another inequality was shaded with blue, then the overlapping area would be shaded purple. The points in the purple area would be the solution set of this system.

Example: Solve by graphing:

$$x + y \leq 6$$

$$x - 2y \leq 6$$

Solving the inequalities for  $y$ , they become:

$$y \leq -x + 6 \quad (y \text{ intercept of } 6 \text{ and slope} = -1)$$