COMPETENCY 1

USE MATHEMATICAL REASONING IN PROBLEM-SOLVING SITUATIONS TO ARRIVE AT LOGICAL CONCLUSIONS AND TO ANALYZE THE PROBLEM-SOLVING PROCESS

SKILL 1.1 Analyze problem solutions for logical flaws

DEDUCTIVE REASONING is the process of arriving at a conclusion based on other statements that are known to be true.

A symbolic argument consists of a set of premises and a conclusion in the format of *if* [premise 1 and premise 2], *then* [conclusion].

An argument is **VALID** when the conclusion follows necessarily from the premises. An argument is **INVALID** or a fallacy when the conclusion does not follow from the premises.

DEDUCTIVE REASONING: the process of arriving at a conclusion based on other statements that are known to be true

VALID: an argument is valid when the conclusion follows necessarily from the premises

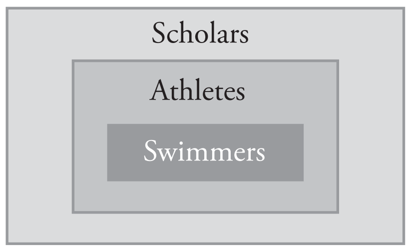
INVALID: an argument is invalid when the conclusion does not follow from the premises

FOUR STANDARD FORMS OF VALID ARGUMENTS		
Law of Detachment	If p, then q p Therefore, q	premise 1 premise 2
Law of Contraposition	If p, then q not q Therefore, not p	premise 1 premise 2
Law of Syllogism	If p, then q If q, then r Therefore, if p, then r	premise 1 premise 2
Disjunctive Syllogism	p or q not p Therefore, q	premise 1 premise 2

Example: Can we reach a conclusion from these two statements?

A. All swimmers are athletes.
All athletes are scholars.

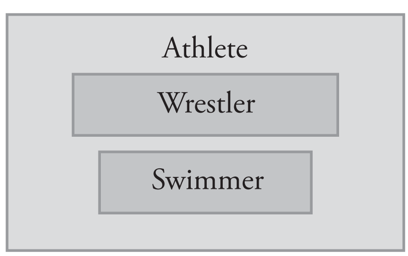
In "if-then" form, these would be:
If you are a swimmer, *then* you are an athlete.
If you are an athlete, *then* you are a scholar.



Clearly, if you are a swimmer, then you are also an athlete. This includes you in the group of scholars.

B. All swimmers are athletes.
All wrestlers are athletes.

In "if-then" form, these would be:
If you are a swimmer, *then* you are an athlete.
If you are a wrestler, *then* you are an athlete.



If you are a swimmer or a wrestler, then you are also an athlete. This does *not* allow you to come to any other conclusions.

A swimmer may or may *not* also be a wrestler. Therefore, *no conclusion is possible*.

Example: Determine whether statement A, B, C, or D can be deduced from the following:

(i) If John drives the big truck, then the shipment will be delivered.
(ii)The shipment will not be delivered.

A. John does not drive the big truck.

B. John drives the big truck.

C. The shipment will not be delivered.

D. None of the above conclusions is valid.

Let p: John drives the big truck.
q: The shipment is delivered.
Statement (i) gives $p \rightarrow q$. Statement (ii) gives $\sim q$ (not q). This is the Law of Contraposition.
Therefore, the logical conclusion is $\sim p$ (not p), or "John does not drive the big truck." The answer is A.

Example: Determine which conclusion can be logically deduced from the following information:

(i) Peter is a jet pilot or Peter is a navigator.
(ii) Peter is not a jet pilot.

A. Peter is not a navigator.

B. Peter is a navigator.

C. Peter is neither a jet pilot nor a navigator.

D. None of the above is true.

Let p: Peter is a jet pilot.
q: Peter is a navigator.
So we have $p \vee q$ (p or q) from statement (i)
$\sim p$ (not p) from statement (ii)
The answer is B.

Example: What conclusion, if any, can be reached? Assume each statement is true, regardless of any personal beliefs.

1. If the Red Sox win the World Series, I will die.
I died.
The Red Sox won the World Series.

2. If an angle's measure is between 0° and 90°, then the angle is acute.
Angle B is not acute.
Angle B is not between 0° and 90°.

3. Students who do well in geometry will succeed in college.
Annie is doing extremely well in geometry.
Annie will do well in college.

4. Left-handed people are witty and charming.
You are left-handed.
You are witty and charming.

SKILL 1.2 Examine problems to determine missing information needed to solve them

Some problems do not contain enough information with which to solve them.

Example: During one semester, a college student used 70 gallons of gas driving back and forth to visit her family. The total cost of gas was $225. What was the average number of gallons of gas used per trip?
This question cannot be answered because you do not know the number of trips the student made.

SKILL 1.3 Analyze a partial solution to a problem to determine an appropriate next step

Example: A fish is 30 inches long. The head is as long as the tail. If the head was twice as long and the tail was the same length, then the body would be 18 inches long. How long is the body?
Partial solution: Let x represent the head.

$$2x + x + 18 = 30$$
$$3x = 12$$
$$x = 4$$

We now create an equation to solve for the body of the fish with y representing the body.

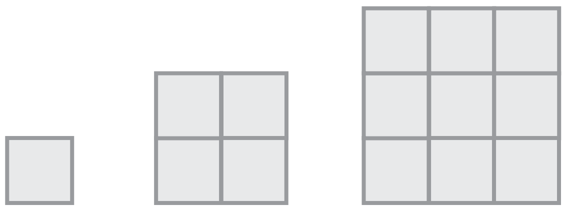
$$x + x + y = 30$$
$$2x + y = 30$$

Substitute 4 for x.

$$2(4) + y = 30$$
$$8 + y = 30$$
$$3y = 30$$
$$y = 22$$

In this example, we are able to substitute the partial solution to solve for the variable in the problem's actual question.

Example: How many squares must be added to a 10-by-10 square to create an 11-by-11 square?



1-square 2-squares 3-squares

Partial solution: We determine that a 3-by-3 square has 5 more squares than a 2-by-2 square, which has 3 more squares than 1 square.

By examining the pattern, we can answer the question by adding the dimension of the previous square (in this case, 10) to the dimension of the current square (in this case, 11). Twenty-one squares must be added to a 10-by-10 square to create an 11-by-11 square.

SKILL 1.4 Evaluate the validity or logic of an argument or advertising claim that is based on statistics or probability

Statistics and probability deal with inherently uncertain situations where it may not be possible to evaluate whether a hypothesis is completely valid or completely invalid. There are mathematical constructs, however, that can be used to assign a certain degree of validity to a statistical result.

Random sampling is the process of studying an aspect of a population by selecting and gathering data from a segment of the population and making inferences and generalizations based on the results. Sample statistics such as mean, median, mode, range, and sampling error (standard deviation) are important generalizations about the entire sample. Various factors affect the accuracy of sample statistics and the generalizations made from them about the larger population. Sample size is one important factor in the accuracy and reliability of sample statistics. As sample size increases, sampling error (standard deviation) decreases.

Sampling error is the main determinant of the size of the **confidence interval**. Confidence intervals decrease in size as sample size increases. A confidence interval gives an estimated range of values, which is likely to include a particular population parameter. The **confidence level** associated with a confidence interval is the probability that the interval contains the population parameter. For example, a poll reports that 60% of a sample group prefers candidate A with a margin of error of ±3% and a confidence level of 95%. In this poll, there is a 95% chance that the preference for candidate A in the whole population is between 57% and 63%.

Apart from mathematical calculations of validity, there are other factors that influence the validity of a statistical result that must be taken into account when deciding whether an argument or advertising claim is valid. One question to ask is what the sampling procedure is and how bias-free it is. If an advertiser claims that a survey of 5,000 people shows that a particular toothpaste is preferred by most people, it makes sense to ask how that sample of 5,000 was selected. Is it an accurate representation of the general population? Exactly what questions were they asked? Is there any inherent bias in the kind of questions asked?

If the results of a study are shown in graphical form, what are the scales used on the axes? Graphical results can look very different with different choices of scales. If an average is quoted, is this average the mean, median or mode? The average of a data set does not include information about the data spread. Is the data spread normal or skewed in some way?

For instance, if the average salary at a company is $40,000, does everyone make an amount that is close to that number or do a few management people earn millions and the rest of the people a lot less than the stated number? If a percentage is quoted, it is necessary to ascertain what that percentage is relative to. For example, if an advertiser claims that the quantity of a bottle of detergent is increased by 33%, this does not mean that the price is reduced by 33%. The necessary calculations may show that price is reduced by only 25%. Other questions to ask include whether the figures given are too precise given the study methods or whether correlation is being confused with causation.

COMPETENCY 2

UNDERSTAND CONNECTIONS BETWEEN MATHEMATICAL REPRESENTATIONS AND IDEAS, AND USE MATHEMATICAL TERMS AND REPRESENTATIONS TO ORGANIZE, INTERPRET, AND COMMUNICATE INFORMATION

SKILL 2.1 Analyze data, and make inferences from two or more graphic sources *(e.g., diagrams, graphs, equations)*

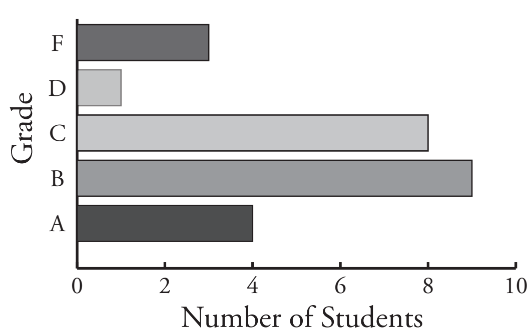
To make a **BAR GRAPH** or a **PICTOGRAPH**, determine the scale to be used for the graph. Then determine the length of each bar on the graph, or determine the number of pictures needed to represent each item of information. Be sure to include an explanation of the scale in the legend.

BAR GRAPH: a graph that compares various quantities

PICTOGRAPH: a graph that compares quantities using symbols where each symbol represents a number of items

Example: A class had the following grades: 4 As, 9 Bs, 8 Cs, 1 D, and 3 Fs. Graph these on a bar graph and a pictograph.

Bar graph



Pictograph

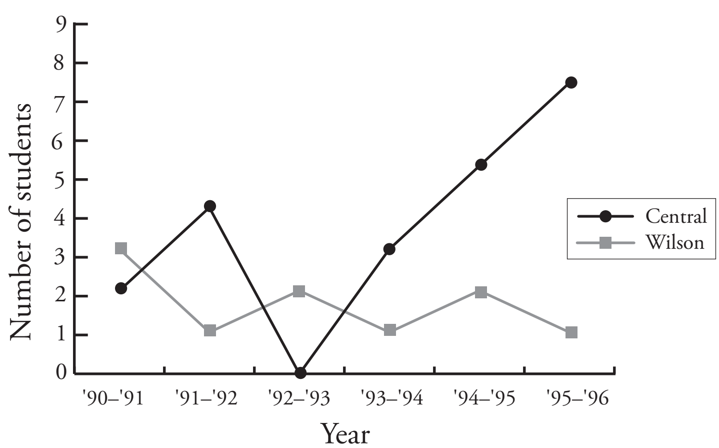
Grade	Number of Students
A	☺☺☺☺
B	☺☺☺☺☺☺☺☺☺
C	☺☺☺☺☺☺☺☺
D	☺
F	☺☺☺

LINE GRAPH: a graph that shows trends, often over a period of time

To make a **LINE GRAPH**, determine appropriate scales for both the vertical and horizontal axes (based on the information to be graphed). Describe what each axis represents, and mark the scale periodically on each axis. Graph the individual points of the graph and connect the points on the graph from left to right.

Example: Graph the following information using a line graph.
The number of National Merit Scholarship finalists per school year

	90–91	91–92	92–93	93–94	94–95	95–96
Central	3	5	1	4	6	8
Wilson	4	2	3	2	3	2



CIRCLE GRAPH: also called a pie chart, this graph shows quantities in proportional sectors

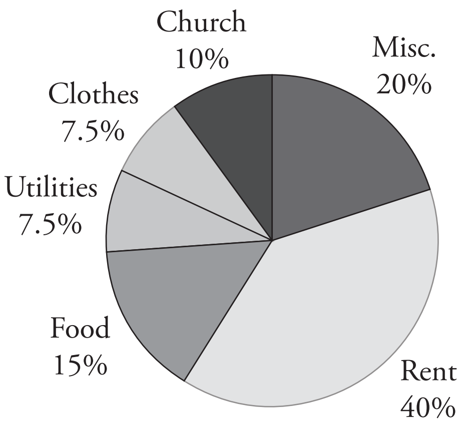
To make a **CIRCLE GRAPH**, total all the information that is to be included on the graph. Determine the central angle to be used for each sector of the graph using the following formula:

$$\frac{\text{information}}{\text{total information}} \times 360^{\circ} = \text{degrees in central} \measuredangle$$

Lay out the central angles to these sizes, label each section, and include each section's percent.

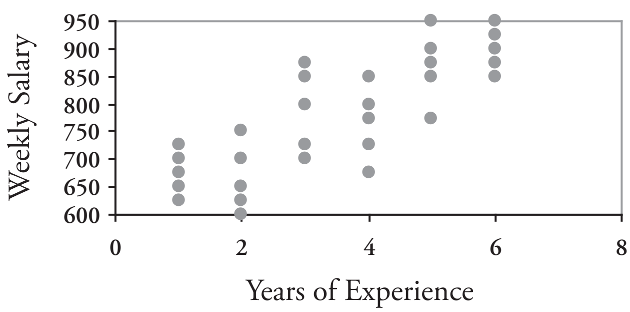
Example: Graph the following information about monthly expenses using a circle graph.

MONTHLY EXPENSES					
Rent	**Food**	**Utilities**	**Clothes**	**Church**	**Misc.**
$400	$150	$75	$75	$100	$200



Scatter plots compare two characteristics of the same group of things or people and usually consist of a large body of data. They show how much one variable affects another. The relationship between the two variables is their **CORRELATION**. The closer the data points come to forming a straight line when plotted, the closer the correlation.

CORRELATION: the relationship between the two variables in a scatter plot



Stem-and-leaf plots are visually similar to line plots. The **stems** are the digits in the greatest place value of the data values, and the **leaves** are the digits in the next greatest place value. Stem-and-leaf plots are best suited to small sets of data and are especially useful for comparing two sets of data. The following is an example using test scores:

4	9
5	4 9
6	1 2 3 4 6 7 8 8
7	0 3 4 6 6 6 6 7 7 7 7 8 8 8 8
8	3 5 5 7 8
9	0 0 3 4 5
10	0 0

HISTOGRAM: a graph that summarizes information from large sets of data that can be naturally grouped into intervals

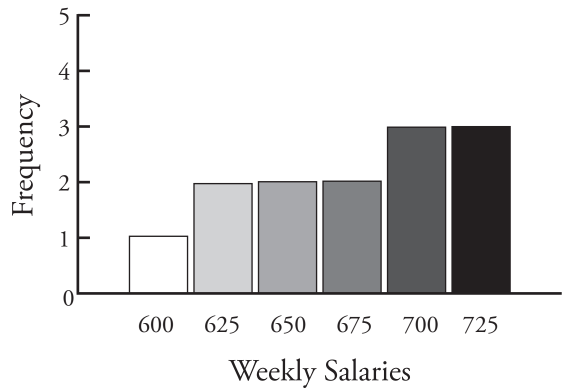
FREQUENCY: the number of times any particular data value occurs

FREQUENCY OF THE INTERVAL: the number of data values in any interval

TREND: a line on a line graph that shows the correlation between two sets of data

INFERENCE: a statement that is derived from reasoning

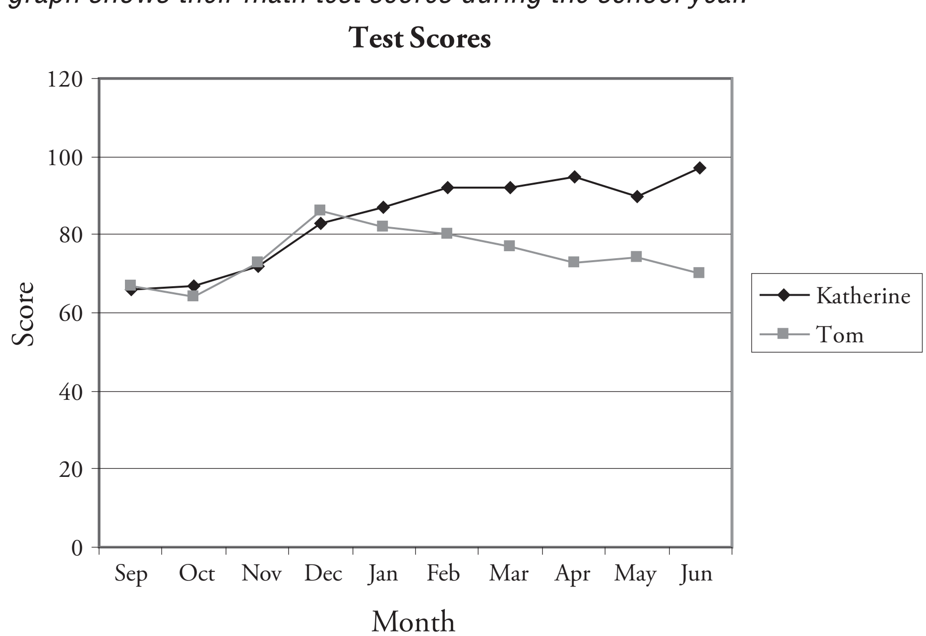
HISTOGRAMS are used to summarize information from large sets of data that can be naturally grouped into intervals. The vertical axis indicates **FREQUENCY** (the number of times any particular data value occurs), and the horizontal axis indicates data values or ranges of data values. The number of data values in any interval is the **FREQUENCY OF THE INTERVAL**.



A **TREND** line on a line graph shows the correlation between two sets of data. A trend may show positive correlation (both sets of data get bigger together), negative correlation (one set of data gets bigger while the other gets smaller), or no correlation.

An **INFERENCE** is a statement that is derived from reasoning. When reading a graph, inferences help us interpret the data that is presented. From this information, a conclusion and even predictions about what the data actually mean are possible.

Example: Katherine and Tom were both doing poorly in math class. Their teacher had a conference with each of them in November. The following graph shows their math test scores during the school year.



What kind of trend does this graph show?

This graph shows that there is a positive trend in Katherine's test scores and a negative trend in Tom's test scores.

What inferences can you make from this graph?

We can infer that Katherine's test scores rose steadily after November. Tom's test scores spiked in December but then began to fall again and trended negatively.

What conclusion can you draw based on this graph?

We can conclude that Katherine took her teacher's meeting seriously and began to study in order to do better on the exams. It seems as though Tom tried harder for a bit, but his test scores eventually slipped back down to the level at which they began.

SKILL 2.2 Restate a problem related to a concrete situation in mathematical terms

Example: The YMCA wants to sell raffle tickets to raise at least $32,000. If they must pay $7,250 in expenses and prizes out of the money collected from the tickets, how many tickets worth $25 each must they sell?

Since they want to raise at least $32,000, that means they would be happy to get $32,000 *or more*. This requires an inequality.

Let x = number of tickets sold.

Then $25x$ = total money collected for x tickets.

Total money minus expenses must be greater than $32,000.

$$25x - 7250 \geq 32000$$

$$25x \geq 39250$$

$$x \geq 1570$$

If they sell 1,570 tickets or more, they will raise at least $32,000.

For more examples see, Skill 3.5.

SKILL 2.3 Use mathematical modeling/multiple representations to present, interpret, communicate, and connect mathematical information and relationships

See Skill 2.1

SKILL 2.4 Select an appropriate graph or table summarizing information presented in another form *(e.g., a newspaper excerpt)*

See Skill 2.1

COMPETENCY 3

APPLY KNOWLEDGE OF NUMERICAL, GEOMETRIC, AND ALGEBRAIC RELATIONSHIPS IN PROBLEM SOLVING AND MATHEMATICAL CONTEXTS

SKILL 3.1 Represent and use numbers in a variety of equivalent forms *(e.g., integer, fraction, decimal, percent)*

RATIONAL NUMBERS: numbers that can be expressed as the ratio of two integers, $\frac{a}{b}$, where b ≠ 0

INTEGERS: the positive and negative whole numbers and zero

WHOLE NUMBERS: the natural numbers and zero

NATURAL NUMBERS: the counting numbers

RATIONAL NUMBERS are numbers that can be expressed as the ratio of two integers, $\frac{a}{b}$, where $b \neq 0$. For example, $\frac{2}{3}$, $-\frac{4}{5}$, and $5 = \frac{5}{1}$ are all rational numbers.

The rational numbers include integers, fractions and mixed numbers, and terminating and repeating decimals. Every rational number can be expressed as a repeating or terminating decimal and can be shown on a number line.

INTEGERS are the positive and negative whole numbers and zero.

...−6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6...

WHOLE NUMBERS are the natural numbers and zero.

0, 1, 2, 3, 4, 5, 6...

NATURAL NUMBERS are the counting numbers.

1, 2, 3, 4, 5, 6...

IRRATIONAL NUMBERS are real numbers that cannot be written as the ratio of two integers. They are infinite, nonrepeating decimals.

$\sqrt{5} = 2.2360$, pi $= \pi = 3.1415927...$

A **FRACTION** is an expression of numbers in the form of $\frac{x}{y}$, where x is the numerator and y is the denominator. The denominator cannot be zero.

$\frac{3}{7}$ 3 is the numerator; 7 is the denominator

If the fraction has common factors in the numerator and denominator, divide both by the common factors to reduce the fraction to its simplest form.

$\frac{13}{39} = \frac{1 \times 13}{3 \times 13} = \frac{1}{3}$

Divide by the common factor 13.

A **MIXED NUMBER** has an integer part and a fractional part.

$2\frac{1}{4}, -5\frac{1}{6}, 7\frac{1}{3}$

PERCENT means per 100 (written with the symbol %). Thus, $10\% = \frac{10}{100} = \frac{1}{10}$.

DECIMAL means deci or part of ten. To find the decimal equivalent of a fraction, use the denominator to divide the numerator, as shown in the following example:

Find the decimal equivalent of $\frac{7}{10}$.
Because 10 cannot divide into 7 evenly,
$\frac{7}{10} = 0.7$

The **EXPONENT FORM** is a shortcut method to write repeated multiplication. The basic form is b^n, where b is called the **BASE** and n is the **EXPONENT.** b and n are real numbers. b^n implies that the base b is multiplied by itself n times.

Examples:

$3^4 = 3 \times 3 \times 3 \times 3 = 81$

$2^3 = 2 \times 2 \times 2 = 8$

$(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

$-2^4 = -(2 \times 2 \times 2 \times 2) = -16$

Caution: The exponent does not affect the sign unless the negative sign is inside the parentheses and the exponent is outside the parentheses.

$(-2)^4$ implies that −2 is multiplied by itself 4 times.

-2^4 implies that 2 is multiplied by itself 4 times, and then the answer becomes. negative.

KEY EXPONENT RULES: FOR *'a'* NONZERO AND *'m'* AND *'n'* REAL NUMBERS	
Product Rule	$a^m \times a^n = a^{(m+n)}$
Quotient Rule	$\frac{a^m}{a^n} = a^{(m-n)}$
Rule of Negative Exponents	$\frac{a^{-m}}{a^{-n}} = \frac{a^n}{a^m}$

IRRATIONAL NUMBERS: real numbers that cannot be written as the ratio of two integers; they are infinite, nonrepeating decimals

FRACTION: an expression of numbers in the form of $\frac{x}{y}$, where x is the numerator and y is the denominator

MIXED NUMBER: a number that has an integer part and a fractional part

PERCENT: means "per 100;" ten percent is 10 parts out of 100

DECIMAL: a number written with a whole-number part, a decimal point, and a decimal part

EXPONENT FORM: a shorthand way of writing repeated multiplication; the basic form is b^n, where b is the *base* and n is the *exponent*

BASE: the number to be multiplied as many times as indicated by the exponent

EXPONENT: tells how many times the base is multiplied by itself

When 10 is raised to any power, the exponent tells the numbers of zeros in the product.

Example:

$10^7 = 10{,}000{,}000$

SCIENTIFIC NOTATION: a convenient method for writing very large and very small numbers

SCIENTIFIC NOTATION is a convenient method for writing very large and very small numbers. It employs two factors. The first factor is a number between 1 and 10. The second factor is a power of 10. This notation is considered "shorthand" for expressing very large numbers (such as the weight of 100 elephants) or very small numbers (such as the weight of an atom in pounds).

Recall that:

10^n	=	Ten multiplied by itself *n* times
10^0	=	Any nonzero number raised to the zero power is 1
10^1	=	10
10^2	=	$10 \times 10 = 100$
10^3	=	$10 \times 10 \times 10 = 1000$
10^{-1}	=	$\frac{1}{10}$ (deci)
10^{-2}	=	$\frac{1}{100}$ (centi)
10^{-3}	=	$\frac{1}{1000}$ (milli)
10^{-6}	=	$\frac{1}{1{,}000{,}000}$ (micro)

Example: Write 46,368,000 in scientific notation.

1. Introduce a decimal point and decimal places.
   $46{,}368{,}000 = 46{,}368{,}000.0000$

2. Make a mark between the two digits that give a number between −9.9 and 9.9.
   $4 \wedge 6{,}368{,}000.0000$

3. Count the number of digit places between the decimal point and the $\wedge$ mark. This number is the *nth* power of ten.
   So, $46{,}368{,}000 = 4.6368 \times 10^7$.