COMPETENCY 1

THE TEACHER UNDERSTANDS THE REAL NUMBER SYSTEM AND ITS STRUCTURE, OPERATIONS, ALGORITHMS, AND REPRESENTATIONS

This competency section reviews some of the fundamental concepts of the real number system, including subsets of the real numbers and their associated properties, various representations of real numbers and operations, and the use of irrational numbers in solving problems.

SKILL 1.1 Understands the concepts of place value, number base, and decimal representations of real numbers

Place Value

In a number, every digit has a face value and a place value. The face values of the digits in the number 3467 are 3, 4, 6, and 7. The place value of a digit depends on its position in the number.

Whole number place value is determined by the relative positions of the digits to the left of the decimal point. Consider the number 792: reading from left to right, the first digit (7) represents the hundreds place. Thus, there are seven sets of 100 in the number 792. The second digit (9) represents the tens place, i. e., nine sets of ten. The last digit (2) represents the ones place.

Decimal place value is determined by the relative positions of the digits to the right of the decimal point. Consider the number 4.873: reading from left to right, the first digit (4) is in the ones place. The first digit after the decimal (8) is in the tenths place and indicates that the number contains eight tenths. The next digit (7) is in the hundredths' place and tells us the number contains seven hundredths. The same pattern applies to the rest of the number, with each successive digit to the right of the decimal point decreasing progressively by powers of ten.

Example: The positions of the digits in the number 12345.6789 indicate the following powers of ten:

10^4	10^3	10^2	10^1	10^0		10^{-1}	10^{-2}	10^{-3}	10^{-4}
1	2	3	4	5	.	6	7	8	9

Number Systems Bases

The standard method of writing numbers is the decimal (or base 10) system, where the digits represent powers of 10. Other bases can also be used to represent a number. The binary (or base 2) system, for instance, uses the powers of 2 (2^0, 2^1, 2^2, and so on) to represent a number. Base 2 uses only the digits 0 and 1.

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DECIMAL BINARY CONVERSION

Decimal	Binary	Place Value
1	1	2^0
2	10	2^1
4	100	2^2
8	1000	2^3

Thus, the number 9 in the base-10 system is equal to 1001 in the base-2 system. Fractions, ratios, and other functions alter in the same way.

Base	Number System
2	Binary
3	Ternary
4	Quaternary
5	Quinary
6	Senary
8	Octal
16	Hexadecimal

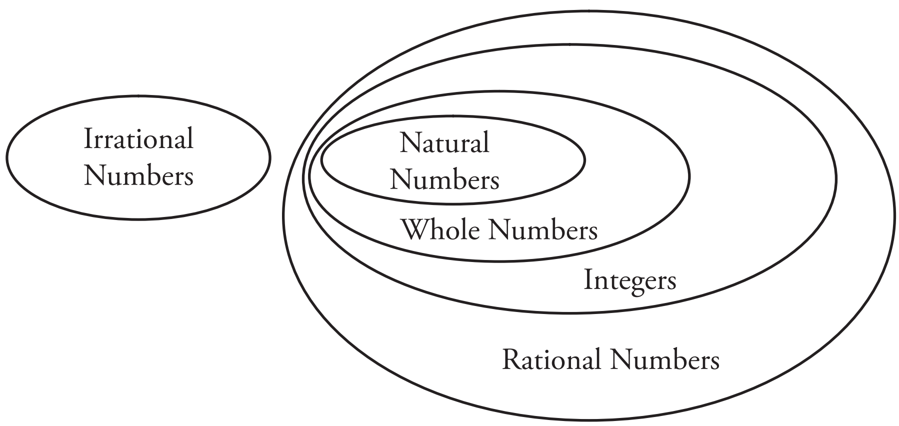
Fundamentally, computers use a binary system, because the transistors and logic circuitry that compose these machines use two basic states (which can be interpreted as 0 and 1, off and on, false and true, or a similar representation). Particular computers may be implemented such that they perform arithmetic on groups of bits (eight bits, for instance, which is called a byte), but the fundamental operation of the machine is still binary.

SKILL 1.2 **Understands the algebraic structure and properties of the real number system and its subsets** *(e.g., real numbers as a field, integers as an additive group)*

Real Numbers

The following chart shows the relationships among the subsets of the real numbers.

Real Numbers



REAL NUMBERS: numbers that can be represented by an infinite decimal representation

REAL NUMBERS are denoted by $\mathbb{R}$ and are numbers that can be shown by an infinite decimal representation such as 3.286275347 . . . . Real numbers include rational numbers, such as 242 and $\frac{-23}{129}$, and irrational numbers, such as $\sqrt{2}$ and π, and can be represented as points along an infinite number line. Real numbers are also known as "the unique complete Archimedean *ordered field*." Real numbers are to be distinguished from imaginary numbers, which involve a factor of $\sqrt{-1}$.

Real numbers are classified as follows:

CLASSIFICATIONS OF REAL NUMBERS	
Natural Numbers Denoted by $\mathbb{N}$	The counting numbers. 1, 2, 3, . . .
Whole Numbers	The counting numbers along with zero. 0, 1, 2, 3, . . .
Integers, Denoted by $\mathbb{Z}$	The counting numbers, their negatives, and zero. . . . , −2, −1, 0, 1, 2, . . .

Continued on next page

Rationals, Denoted by $\mathbb{Q}$	All of the fractions that can be formed using whole numbers. Zero cannot be the denominator. In decimal form, these numbers will be either terminating or repeating decimals. Simplify square roots to determine if the number can be written as a fraction.
Irrationals	Real numbers that cannot be written as a fraction. The decimal forms of these numbers neither terminate nor repeat. Examples include π, e and $\sqrt{2}$.

SKILL 1.3 **Describes and analyzes properties of subsets of the real numbers** *(e.g., closure, identities)*

Fields, Rings, and Groups

FIELD: any set that includes at least two nonzero elements that satisfies the field axioms for addition and multiplication

Any set that includes at least two nonzero elements that satisfies the field axioms for addition and multiplication is a **FIELD**. The real numbers, $\mathbb{R}$, as well as the complex numbers, $\mathbb{C}$, are each a field, with the real numbers being a subset of the complex numbers. The field axioms are summarized below.

FIELD AXIOMS	
ADDITION	
Commutativity	$a + b = b + a$
Associativity	$a + (b + c) = (a + b) + c$
Identity	$a + 0 = a$
Inverse	$a + (-a) = 0$
MULTIPLICATION	
Commutativity	$ab = ba$
Associativity	$a(bc) = (ab)c$
Identity	$a \times 1 = a$
Inverse	$a \times \frac{1}{a} = 1 \qquad (a \neq 0)$
ADDITION AND MULTIPLICATION	
Distributivity	$a(b + c) = (b + c)a = ab + ac$

Note that both the real numbers and the complex numbers satisfy the axioms summarized above.

A **RING** is an integral domain with two binary operations (addition and multiplication) where, for every nonzero element a and b in the domain, the product ab is nonzero. A field is a ring in which multiplication is commutative, or $a \times b = b \times a$, and all nonzero elements have a multiplicative inverse. The set $\mathbb{Z}$ (integers) is a ring that is not a field in that it does not have the multiplicative inverse; therefore, integers are not a field. A polynomial ring is also not a field, as it also has no multiplicative inverse. Furthermore, matrix rings do not constitute fields because matrix multiplication is not generally commutative.

RING: an integral domain with two binary operations (addition and multiplication) where, for every nonzero element *a* and *b* in the domain, the product *ab* is nonzero

Note: *Multiplication is implied when there is no symbol between two variables. Thus, a × b can be written ab. Multiplication can also be indicated by a raised dot (·).*

A **GROUP** is a set of numbers that obeys certain axioms with respect to a particular binary operation (such as addition). For a set, G, to be a group, the set must be closed under the defined operation, which must obey associativity; the set must contain an identity element; and each element must have an inverse element also in set G. These rules are summarized below for elements a, b, and c in set G for the binary operation *.

GROUP: a set of numbers that obeys certain axioms with respect to a particular binary operation (such as addition)

PROPERTY	NOTATION
Closure	$a * b \in G$
Associativity	$a * (b * c) = (a * b) * c$
Identity	$I \in G$ such that $I * a = a * I = a$
Inverse	$a_{inv} \in G$ such that $a_{inv} * a = a * a_{inv} = I$

According to the inverse rule, I is the same as the identity element. An example of a group is the set of integers under addition. For any two integers a and b, the sum $a + b$ is also an integer—thus, the set of integers is closed under addition. Also, associativity applies, since $a + (b + c) = (a + b) + c$ for any integers a, b, and c. The identity element 0 (zero) is also in the set of integers, and $0 + a = a + 0 = a$, for all a. Furthermore, the inverse element $-a$, for all a, leads to $a + (-a) = 0$. Thus, the set of integers also obeys the identity and inverse axioms, meaning that integers are a group under the binary operation of addition.

Real numbers are an ordered field and can be ordered

As such, an ordered field F must contain a subset P (such as the positive numbers) such that if a and b are elements of P, then both $a + b$ and ab are also elements of P. (In other words, the set P is closed under addition and multiplication.) Furthermore, it must be the case that for any element c contained in F, exactly one of the following conditions is true: c is an element of P, $-c$ is an element of P, or $c = 0$.

The rational numbers also constitute an ordered field

The set P can be defined as the positive rational numbers. For each a and b that are elements of the set $\mathbb{Q}$ (the rational numbers), $a + b$ is also an element of P, as is ab. (The sum $a + b$ and the product ab are both rational if a and b are rational.) Since P is closed under addition and multiplication, $\mathbb{Q}$ constitutes an ordered field.

SKILL 1.4 Selects and uses appropriate representations of real numbers *(e.g., fractions, decimals, percents, roots, exponents, scientific notation)* for particular situations

Real numbers can be represented in a variety of formats. Some of these formats are more amenable to certain types of problems than others, and it is important to be able to select the proper representation of a real number for a given situation.

For instance, if exact calculations are required, decimal representations (or, similarly, percent representations) of irrational numbers are not appropriate. The use of a decimal necessarily requires use of a finite representation; thus, the decimal form of an irrational number must be rounded to some digit, leading to inaccuracies in calculations. Therefore, irrational numbers such as π and the square roots of certain integers should often be left in their symbolic or square root forms. If inexact calculations are acceptable, a decimal or approximate fractional representation may be suitable.

If the decimal is repeating (such as 0.1111111…), a fractional representation may be the best approach. A fraction can be manipulated easily, and it is sometimes more conducive to exact calculations than are repeating decimals (or even long non-repeating decimals in some cases).

In other instances, an exponential form is useful. Exponentials (or their inverses, logarithms) may be a preferred representation of real numbers in various cases. Whether exact or inexact calculations are needed for a particular problem, the simplicity of the calculation is also important when selecting an appropriate representation of a number. For hand or mental calculations, simplicity may be paramount, for instance.

SCIENTIFIC NOTATION: a convenient method for writing very large and very small numbers

SCIENTIFIC NOTATION is a convenient method for writing very large and very small numbers. It employs two factors: the first factor is a number between 1 and 9, inclusive, and the second factor is a power of 10. This notation is a shorthand way to express very large numbers (such as the weight in kilograms of 100 freight cars) or very small numbers (such as the weight in grams of an atom).

For example, 356.73 can be written in various forms.

$$356.73 = 3567.3 \times 10^{-1} \quad (1)$$
$$= 35673 \times 10^{-2} \quad (2)$$
$$= 35.673 \times 10^{1} \quad (3)$$
$$= 3.5673 \times 10^{2} \quad (4)$$
$$= 0.35673 \times 10^{3} \quad (5)$$

Only line 4 is proper scientific notation.

Example: Write 46,368,000 in scientific notation.

1. Introduce a decimal point. 46,368,000 = 46,368,000.0
2. Move the decimal point to the left until only one nonzero digit precedes it (in this case, the decimal point is placed between the 4 and the 6).
3. Count the number of digits the decimal point moved (in this case, seven). This is the n^{th} power of 10 and is positive because the decimal point moved to the left.

   Therefore, $46{,}368{,}000 = 4.6368 \times 10^{7}$.

Example: Write 0.00397 in scientific notation.

1. The decimal point is already in place.
2. Move the decimal point to the right until there is only one nonzero digit in front of it (in this case, the decimal point is placed between the 3 and the 9).
3. Count the number of digits the decimal point moved (in this case, three). This is the n^{th} power of ten and is negative because the decimal point moved to the right.

   Therefore, $0.00397 = 3.97 \times 10^{-3}$.

   Thus, there are a number of possible representations for a given real number depending on the problem or situation under consideration. The following examples illustrate some problems where certain representations of given real numbers are better than others.

Example: A particular material has a mass of 0.01 grams in one liter. What is the material's density in grams per milliliter?

A cursory examination of this problem shows that it will be necessary to divide a small number (0.01 grams) by a large number (1000 milliliters = 1 liter) to get the density. Scientific notation is a helpful representation of the numbers in the problem. The density d is represented as

$$d = \frac{1 \times 10^{-2}\ \text{g}}{1 \times 10^{3}\ \text{mL}} = 1 \times 10^{-5}\ \frac{\text{g}}{\text{mL}}.$$

The calculation and the result in this case are simplified considerably through the use of scientific notation. The solution is in a much neater form than 0.00001.

Example: Express the repeating decimal $0.\overline{254}$ as a number in closed form.
This problem calls for selecting an appropriate closed-form representation in the real number system for a repeating decimal. First, note that because the decimal repeats, the three repeating digits can be isolated as follows. Let *d* be equal to the repeating decimal $0.\overline{254}$.

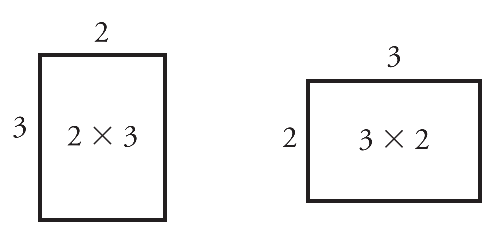
$$1000d = 254.\overline{254} = 254 + d$$
$$999d = 254$$
$$d = \frac{254}{999}$$

Thus, this repeating decimal can be expressed in closed form using a fractional representation.

SKILL 1.5 Uses a variety of models *(e.g., geometric, symbolic)* to represent operations, algorithms, and real numbers

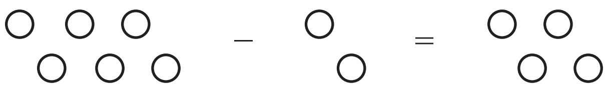
Operations

Number operations can be represented in various ways (for instance, to aid understanding of the operation or to simplify the solution of a particular problem). Multiplication of two numbers, for example, can be represented geometrically as the area of a rectangle with width and length equal to the two factors. Consider the operation 2×3.

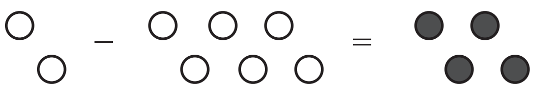


Notice that these two rectangles are identical, and therefore 2×3 is the same as 3×2. This fact is another way to understand the commutativity of multiplication for the set of real (and complex) numbers.

Addition and subtraction can also be expressed using arbitrary symbols to represent integral (or fractional) values. The operation $6 - 2$ can be represented as shown below using circles.



In such a case, negative numbers could be represented as filled circles (with a negative value conceptually corresponding, for example, to "debt"). The symbolic representation below illustrates the operation $2 - 6$.



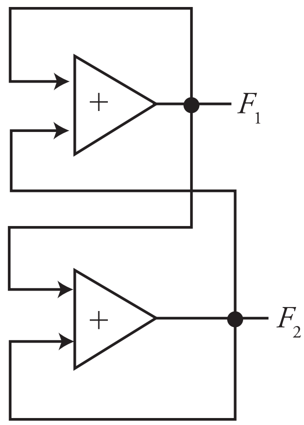
Algorithms

Likewise, algorithms can be represented in various ways. Algorithms intended to run on a computer can be written in a specific programming language (such as FORTRAN, C, or Java) or in "pseudocode," which is an unspecified approach to writing algorithms in a manner that is similar to, but more general than, specified programming languages. A simple example is an algorithm that generates the Fibonacci numbers. The following "pseudocode" illustrates such an algorithm for generating the first N Fibonacci numbers.

$F_i \leftarrow 0$
$F_{i+1} \leftarrow 0$
If $N \geq 0$, Output F_i
If $N \geq 1$, Output F_{i+1}
$i \leftarrow 2$
While $i \geq N$:
    $F_{i+2} \leftarrow F_i + F_{i+1}$
Output F_{i+2}
$F_i \leftarrow F_{i+1}$
    $F_{i+1} \leftarrow F_{i+2}$
    $i \leftarrow i + 1$
    End

This algorithm "outputs" the first N Fibonacci numbers regardless of the value chosen for N (assuming N is an integer greater than or equal to zero).

An algorithm can also be represented using a block diagram or flow chart. The diagram below illustrates the Fibonacci algorithm.



If F_1 and F_2 are initialized to 0 and 1, respectively, this diagram represents a recursive algorithm that produces the Fibonacci numbers (specifically, if F_1 is followed). The triangular blocks represent summation of the inputs. Each time the system is "updated," the next successive Fibonacci number is produced (the full sequence is only produced at the F_1 output).

SKILL 1.6 Uses real numbers to model and solve a variety of problems

The set of real numbers is a crucial component of innumerable mathematical problems. Selection of the appropriate subset and representation of the real numbers, as well as accurate calculation in accordance with the characteristics of that subset, are critical to correctly modeling and solving problems.

The set of real numbers is a crucial component of innumerable mathematical problems. Selection of the appropriate subset and representation of the real numbers, as well as accurate calculation in accordance with the characteristics of that subset, are critical to correctly modeling and solving problems. The other skill sections in this competency, as well as numerous other sections throughout the guide, provide example problems that illustrate the proper use of real numbers in modeling and solving problems in a variety of mathematical contexts. The example problem below also illustrates the use of real numbers.

Example: If a, b, and c are positive real numbers, prove that c(a + b) = (b + a)c.

Use the properties of the set of real numbers.

$c(a+b)$	$= c(b+a)$	Additive commutativity
	$= cb + ca$	Distributivity
	$= bc + ac$	Multiplicative commutativity
	$= (b+a)c$	Distributivity

SKILL 1.7 Uses deductive reasoning to simplify and justify algebraic processes

The properties of real and complex numbers (see Competency 002) can be applied to the construction of various mathematical arguments. A mathematical argument proves that a proposition is true (or false). Deductive reasoning involves making particular inferences based on general premises or axioms. Application of these premises and the rules of deductive logic can be very helpful when solving problems. The example problems below illustrate the use of some of the fundamental principles of basic deductive logic (such as syllogisms).

Example: Prove that for every integer y, if y is an even number, then y^2 is even.

The definition of *even* implies that for each integer y there is at least one integer x such that $y = 2x$.

$$y = 2x$$
$$y^2 = 4x^2$$

Since $4x^2$ is always evenly divisible by two ($2x^2$ is an integer), y^2 is even for all values of y.

Example: Given real numbers a, b, c, and d, where ad = −bc, prove that (a + bi)(c + di) is real.

Expand the product of the complex numbers.

$$(a + bi)(c + di) = ac + bci + adi + bdi^2$$

Use the definition of i^2.

$$(a + bi)(c + di) = ac - bd + bci + adi$$

Apply the fact that $ad = -bc$.

$$(a + bi)(c + di) = ac - bd + bci - bci = ac - bd$$

Since a, b, c and d are all real, $ac - bd$ must also be real.

Example: Determine if the set of irrational numbers is closed under addition.

One option for solving this problem is to attempt to prove conclusively the positive assertion that the set is closed under addition. An alternative, however, is to attempt to prove the opposite of an assertion through counterexample. In this case, attempt to find two irrational numbers whose sum is not irrational. Consider, for instance, $\sqrt{2}$ and $-\sqrt{2}$.

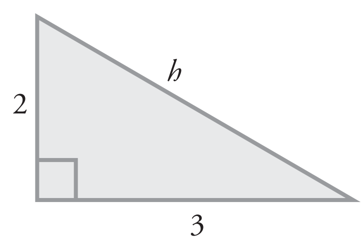
$$\sqrt{2} + (-\sqrt{2}) = 0$$

Since 0 is a rational number in this counterexample, the set of irrational numbers is not closed under addition.

SKILL 1.8 Demonstrates how some problems that have no solution in the integer or rational number systems have solutions in the real number system

The set of real numbers is composed entirely of the union of two mutually exclusive sets: the set of rational numbers and the set of irrational numbers. In many cases, the solutions to certain problems may not be found among the rational numbers, yet the solutions are indeed real numbers. Consider the following examples.

Example: Find the hypotenuse of a right triangle with legs of length 2 and 3.
The diagram below illustrates the problem.



The length h of the hypotenuse can be found using the Pythagorean theorem as follows:

$$h^2 = 2^2 + 3^2 = 4 + 9 = 13$$
$$h = \sqrt{13}$$

The result, $\sqrt{13}$, is not a rational number. Thus, in this case, although the length h is obviously in the set of real numbers, it is not in the set of rational numbers. Logically, the result must be an irrational number, as calculated above.

Example: Calculate the volume of a circular cylinder with a diameter of 4 centimeters and a height of 10 centimeters.
The formula for the volume of a circular cylinder is $\pi r^2 h$. In this case, the radius r is 2 cm and the height h is 10cm. The volume is

$$V = \pi r^2 h = \pi(2)^2\,(10) = 40\pi.$$

Since π is an irrational number, the product 40π is likewise irrational. Therefore, although the volume of the cylinder is not an integer or otherwise a rational number, it is a real number.

COMPETENCY 2

THE TEACHER UNDERSTANDS THE COMPLEX NUMBER SYSTEM AND ITS STRUCTURE, OPERATIONS, ALGORITHMS, AND REPRESENTATIONS

The set of complex numbers includes all real numbers but is expanded through use of the factor $i = \sqrt{-1}$. This section reviews the fundamental concepts and applications of complex numbers, including operations, representations of numbers and operations, and problems involving the complex domain.

SKILL 2.1 Demonstrates how some problems that have no solution in the real number system have solutions in the complex number system

Numerous problems do not have solutions that are contained in the set of real numbers. In such cases, complex solutions may be required. Consider the following canonical example.

$$x^2 + 1 = 0$$

This equation cannot be solved for x in the real domain.

$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

Since there is no real number whose square is −1, there is no real solution for x. The imaginary number i is a solution, however; thus, this equation can be solved in the complex domain.

$$x = \pm i$$

More generally, the Fundamental Theorem of Algebra states that any polynomial of degree n must have n solutions (or roots), which may include real, complex, or nondistinct roots (or some combination thereof).

Other operations can likewise lead to imaginary results. Consider, for instance, the natural logarithm of a negative numbers (for more on logarithms, see Competency 008).

$$\ln(-1) = ?$$

In the real domain, this expression is undefined, because there is no real exponent of e that results in a negative number. If Euler's formula is applied, however, a solution from the set of complex numbers becomes apparent.

$$\ln(-1) = \ln(\cos \pi + i \sin \pi) = \ln(e^{i\pi})$$
$$\ln(-1) = i\,\pi \ln(e) = i\,\pi$$

Thus, the expression has an equivalent numerical expression from the set of complex numbers. (For precision, it is noteworthy that the general result is in π, where n is an integer.)

Example: Find all the roots of the polynomial $x^3 - x^2 + 3x - 3$.

This problem can be tackled in any of several ways. First, note that the Fundamental Theorem of Algebra requires that the polynomial must have three solutions. In this case, the polynomial can be factored. By inspection, it appears that $x = 1$ might be a solution:

$$(1)^3 - (1)^2 + 3(1) - 3 = 1 - 1 + 3 - 3 = 0$$

Thus, $(x - 1)$ must be a factor in the expression. The remaining factor must be a polynomial of degree two.

$$x^3 - x^2 + 3x - 3 = (x - 1)(ax^2 + bx + c)$$

Obviously, a must be equal to 1. Likewise, c must be equal to 3.

$$(x - 1)(x^2 + bx + 3) = x^3 - x^2 + bx^2 - bx + 3x - 3$$
$$x^3 - x^2 + 3x - 3 = x^3 - (1 - b)\,x^2 - (b - 3)\,x - 3$$

Thus, $b = 0$.

$$x^3 - x^2 + 3x - 3 = (x - 1)(x^2 + 3)$$

Factoring once more yields

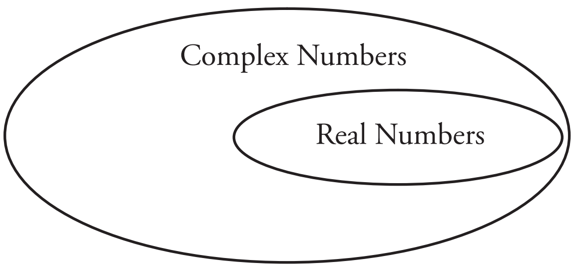
$$0 = (x - 1)(x + i\sqrt{3})(x - i\sqrt{3}).$$

The roots are 1, $i\sqrt{3}$, and $-i\sqrt{3}$.

SKILL 2.2 Understands the properties of complex numbers *(e.g., complex conjugate, magnitude/modulus, multiplicative inverse)*

COMPLEX NUMBERS: numbers of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$

The set of complex numbers is denoted by $\mathbb{C}$. The set $\mathbb{C}$ is defined as $\{a + bi : a, b \in \mathbb{R}\}$ ($\in$ means "element of"). In other words, complex numbers are an extension of real numbers made by attaching an imaginary number i, which satisfies the equality $i^2 = -1$. **COMPLEX NUMBERS** are of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Thus, a is the real part of the number and b is the imaginary part of the number. When i appears in a fraction, the fraction is usually simplified so that i is not in the denominator. The set of complex numbers includes the set of real numbers, where any real number n can be written in its equivalent complex form as $n + 0i$. In other words, it can be said that $\mathbb{R} \subseteq \mathbb{C}$ (or $\mathbb{R}$ is a subset of $\mathbb{C}$).



The number $3i$ has a real part 0 and imaginary part 3; the number 4 has a real part 4 and an imaginary part 0. As another way of writing complex numbers, we can express them as ordered pairs:

Complex Number	Ordered Pair
$3 + 2i$	(3, 2)
$\sqrt{3} + \sqrt{3}i$	$(\sqrt{3}, \sqrt{3})$
$7i$	(0, 7)
$\frac{6 + 2i}{7}$	$\left(\frac{6}{7}, \frac{2}{7}\right)$

The basic operations for complex numbers can be summarized as follows, where $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Note that the operations are performed in the standard manner, where i is treated as a standard radical value. The result of each operation is written in the standard form for complex numbers. Also note that the **COMPLEX CONJUGATE** of a complex number $z = a + bi$ is denoted as $z^* = a - bi$.

COMPLEX CONJUGATE: for a complex number $z = a + bi$, this is denoted as $z^* = a - bi$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$
$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$$
$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 - a_2 b_1)i$$
$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\, i$$

Note that, because the division operation above is defined, the multiplicative inverse of any complex number $z \neq 0$ is also defined (where z_1 is 1 and z_2 is z) in the set of complex numbers.