COMPETENCY 0001 UNDERSTAND NUMBER OPERATIONS AND BASIC PRINCIPLES OF NUMBER THEORY.

The **absolute value** of a real number is the positive value of that number.

$|x|$ = *x* when $x \geq 0$ and

$|x|$ = *x* when $x < 0$.

Example: $|y-7| = 2$

$y - 7 = 2$ *or* $y - 7 = -2$

$y = 9$ *or* $y = 5$

The solutions must be checked.

$\|y-7\| = 2$	$\|y-7\| = 2$
$\|9-7\|$? 2	$\|5-7\|$? 2
$\|2\|$? 2	$\|-2\|$? 2
2 = 2	2 = 2
true	true

Identify elements and subsets of the real number system.

a. **Natural numbers**--the counting numbers, $1, 2, 3, ...$

b. **Whole numbers**--the counting numbers along with zero, $0, 1, 2...$

c. **Integers**--the counting numbers, their opposites, and zero, $..., {}^{-}1, 0, 1, ...$

d. **Rationals**--all of the fractions that can be formed from the whole numbers. Zero cannot be the denominator. In decimal form, these numbers will either be terminating or repeating decimals. Simplify square roots to determine if the number can be written as a fraction.

e. **Irrationals**--real numbers that cannot be written as a fraction. The decimal forms of these numbers are neither terminating nor repeating. Examples: $\pi, e, \sqrt{2}$, etc.

f. **Real numbers**--the set of numbers obtained by combining the rationals and irrationals. Complex numbers, i.e. numbers that involve i or $\sqrt{{}^{-}1}$, are not real numbers.

Compare the relative size of real numbers expressed in a variety of forms, including fractions, decimals, percents, and scientific notation.

To convert a fraction to a decimal, simply divide the numerator (top) by the denominator (bottom). Use long division if necessary.

If a decimal has a fixed number of digits, the decimal is said to be terminating. To write such a decimal as a fraction, first determine what place value the farthest right digit is in, for example: tenths, hundredths, thousandths, ten thousandths, hundred thousands, etc. Then drop the decimal and place the string of digits over the number given by the place value.

If a decimal continues forever by repeating a string of digits, the decimal is said to be repeating. To write a repeating decimal as a fraction, follow these steps.

a. Let $x =$ the repeating decimal
(ex. $x = .716716716...$)
b. Multiply x by the multiple of ten that will move the decimal just to the right of the repeating block of digits.
(ex. $1000x = 716.716716...$)
c. Subtract the first equation from the second.
(ex. $1000x - x = 716.716.716... - .716716...$)
d. Simplify and solve this equation. The repeating block of digits will subtract out.
(ex. $999x = 716$ so $x = 716/999$)
e. The solution will be the fraction for the repeating decimal.

Identify the greatest common factor (GCF) and least common multiple (LCM) of sets of numbers.

GCF is the abbreviation for the **greatest common factor**. The GCF is the largest number that is a factor of all the numbers given in a problem. The GCF can be no larger than the smallest number given in the problem. If no other number is a common factor, then the GCF will be the number 1. To find the GCF, list all possible factors of the smallest number given (include the number itself). Starting with the largest factor (which is the number itself), determine if it is also a factor of all the other given numbers. If so, that is the GCF. If that factor doesn't work, try the same method on the next smaller factor. Continue until a common factor is found. That is the GCF. Note: There can be other common factors besides the GCF.

Example: Find the GCF of 12, 20, and 36.

The smallest number in the problem is 12. The factors of 12 are 1,2,3,4,6 and 12. 12 is the largest factor, but it does not divide evenly into 20. Neither does 6, but 4 will divide into both 20 and 36 evenly.
Therefore, 4 is the GCF.

Example: Find the GCF of 14 and 15.

Factors of 14 are 1,2,7 and 14. 14 is the largest factor, but it does not divide evenly into 15. Neither does 7 or 2. Therefore, the only factor common to both 14 and 15 is the number 1, the GCF.

LCM is the abbreviation for **least common multiple**. The least common multiple of a group of numbers is the smallest number that all of the given numbers will divide into. The least common multiple will always be the largest of the given numbers or a multiple of the largest number.

Example: Find the LCM of 20, 30 and 40.

The largest number given is 40, but 30 will not divide evenly into 40. The next multiple of 40 is 80 (2 x 40), but 30 will not divide evenly into 80 either. The next multiple of 40 is 120. 120 is divisible by both 20 and 30, so 120 is the LCM (least common multiple).

- The Fundamental Theorem of Arithmetic states that every composite (non-prime) number can be written as a product of primes in one, and only one way.

  Prime numbers are whole numbers greater than 1 that have only 2 factors, 1 and the number itself. Examples of prime numbers are 2,3,5,7,11,13,17, or 19. Note that 2 is the only even prime number. When factoring into prime factors, all the factors must be numbers that cannot be factored again (without using 1). Initially numbers can be factored into any 2 factors. Check each resulting factor to see if it can be factored again. Continue factoring until all remaining factors are prime. This is the list of prime factors. Regardless of what way the original number was factored, the final list of prime factors will always be the same.

Remember that the number 1 is neither prime nor composite.

Example: Factor 30 into prime factors.

Factor 30 into any 2 factors.

$5 \cdot 6$	Now factor the 6.
$5 \cdot 2 \cdot 3$	These are all prime factors.
Factor 30	Into any 2 factors.
$3 \cdot 10$	Now factor the 10.
$3 \cdot 2 \cdot 5$	These are the same prime factors even though the original factors were different.

Example: Factor 240 into prime factors.

Factor 240	into any 2 factors.
$24 \cdot 10$	Now factor both 24 and 10.
$4 \cdot 6 \cdot 2 \cdot 5$	Now factor both 4 and 6.
$2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 5$	These are prime factors.

This can also be written as $2^4 \cdot 3 \cdot 5$.

COMPETENCY 0002 UNDERSTAND THE REAL AND COMPLEX NUMBER SYSTEMS.

Real numbers exhibit the following addition and multiplication **properties**, where *a, b*, and *c* are real numbers.

Note: Multiplication is implied when there is no symbol between two variables. Thus, $a \times b$ can be written ab. Multiplication can also be indicated by a raised dot $\cdot$

Closure
For all real numbers *a* and *b*,
$a + b$ is a unique real number.
ab is a unique real number.

Example: Since 2 and 5 are both real numbers, 7 is also a real number.

Example: Since 3 and 4 are both real numbers, 12 is also a real number.

Commutative
For all real numbers *a* and *b,*
$a + b = b + a.$
$ab = ba.$

Example: $5 + {}^{-}8 = {}^{-}8 + 5 = {}^{-}3$

Example: ${}^{-}2 \times 6 = 6 \times {}^{-}2 = {}^{-}12$

Associative
For all real numbers *a, b,* and *c,*
$(a + b) + c = a + (b + c).$
$(ab)c = a(bc).$

Example: $({}^{-}2 + 7) + 5 = {}^{-}2 + (7 + 5)$
$5 + 5 = {}^{-}2 + 12 = 10$

Additive Identity (Property of Zero)
There exists a unique real number 0 (zero) such that
$a + 0 = 0 + a = a$ for every real number *a.*

Example: $17 + 0 = 17$
The sum of any number and zero is that number.

Multiplicative Identity (Property of One)
There exists a unique nonzero real number 1 (one) such that $1 \cdot a = a \cdot 1 = a$.

$a \cdot 1 = a$

<u>Example</u>: $^{-}34 \times 1 = {}^{-}34$
The product of any number and one is that number.

Additive Inverse (Property of Opposites)
For each real number *a*, there exists a real number –*a* (the opposite of *a*) such that the sum of any number and its opposite is zero. $a + (-a) = (-a) + a = 0$.

<u>Example</u>: $25 + {}^{-}25 = 0$

Multiplicative Inverse (Property of Reciprocals)
For each nonzero real number, there exists a real number 1/*a* (the reciprocal of a) such that $a(1/a) = (1/a)a = 1$.

<u>Example</u>: $5 \times \frac{1}{5} = 1$

The product of any number and its reciprocal is one.

Distributive

$a\,(b + c) = ab + ac$

<u>Example</u>: $6 \times ({}^{-}4 + 9) = (6 \times {}^{-}4) + (6 \times 9)$
$6 \times 5 = {}^{-}24 + 54 = 30$

To multiply a sum by a number, multiply each addend by the number, then add the products.

Review of Law of Exponents: If a and b are real numbers and m and n are rational numbers, then,

1. $a^m \times a^n = a^{(m+n)}$

2. $\dfrac{a^m}{a^n} = a^{(m-n)}$

3. $(a^m)^n = a^{(mn)}$

4. $(ab)^m = a^m b^m$

5. $a^{-n} = \dfrac{1}{a^n} = (1/a)^n$

6. If a is any nonzero number, then $a^0 = 1$.

7. $a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}$

8. $\sqrt[n]{a^m}$ (radical form) $= a^{m/n}$ in exponential form.

Rules for operating with exponents:

1. To multiply two or more numbers that have the same base, we add the exponents and keep the same base.

$$2^3 \times 2^2 = 2^5 = 32$$

2. To divide two numbers that have the same base, we subtract the exponents and keep the same base.

$$\frac{3^4}{3^2} = 3^2 = 9$$

3. To change the sign of an exponent, the reciprocal of the number is used.

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{4}{3} \times \frac{4}{3} = \frac{16}{9}$$

4. The answer to any real number raised to a 0 exponent is 1.

$$3^0 = 1$$

$$(^-2)^0 = 1$$

$$^-3^0 = ^-1$$

5. Raising a number with an exponent to another exponent requires multiplying the exponents and keeping the same base.

$$(2^2)^3 = 2^6 = 64$$

Caution: Rules for exponents do not apply unless multiplying or dividing numbers with the same base.

Example 1:

$5^4 + 3^3$ Addition and bases are different, so multiply each number separately and combine the answers.

$$(5 \times 5 \times 5 \times 5) + (3 \times 3 \times 3) = 625 + 27 = 652$$

Distinguish relationships between the complex number system and its subsystems.

Complex numbers are of the form $a + b\,\mathbf{i}$, where a and b are real numbers and $\mathbf{i} = \sqrt{^{-}1}$. When **i** appears in an answer, it is acceptable unless it is in a denominator. When $\mathbf{i^2}$ appears in a problem, it is always replaced by a $^{-}1$. Remember, $\mathbf{i^2} = {^{-}}\mathbf{1}$.

To add or subtract complex numbers, add or subtract the real parts then add or subtract the imaginary parts and keep the **i** (just like combining like terms).

<u>Examples</u>: Add $(2 + 3i) + (^{-}7 - 4i)$.

$2 + {^{-}7} = {^{-}5}$ $\quad$ $3i + {^{-}4i} = {^{-}i}$ so,

$(2 + 3i) + (^{-}7 - 4i) = {^{-}5} - i$

Subtract $(8 - 5i) - (^{-}3 + 7i)$

$8 - 5i + 3 - 7i = 11 - 12i$

To multiply 2 complex numbers, F.O.I.L. the 2 numbers together. Replace $\mathbf{i^2}$ with a ${^{-}}\mathbf{1}$ and finish combining like terms. Answers should have the form $a + b\mathbf{i}$.

Example: Multiply $(8+3i)(6-2i)$ F.O.I.L. this.

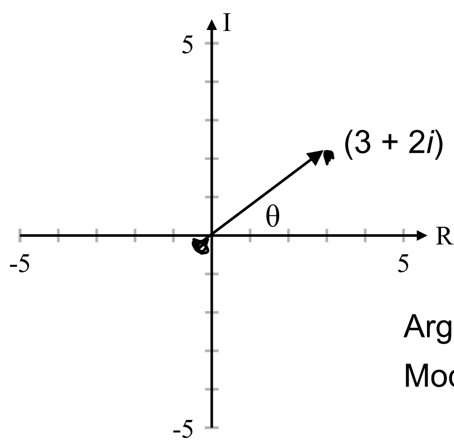
When dividing 2 complex numbers, you must eliminate the complex number in the denominator.
If the complex number in the denominator is of the form *b* **i**, multiply both the numerator and denominator by **i**. Remember to replace $\mathbf{i^2}$ with *a* $^{-}$**1** and then continue simplifying the fraction.

Example:

$$\frac{2+3i}{5i} \qquad \text{Multiply this by } \frac{i}{i}$$

$$\frac{2+3i}{5i}\times\frac{i}{i}=\frac{(2+3i)\,i}{5i\cdot i}=\frac{2i+3i^2}{5i^2}=\frac{2i+3(^{-}1)}{^{-}5}=\frac{^{-}3+2i}{^{-}5}=\frac{3-2i}{5}$$

Complex numbers are numbers of the form *a* + *bi,* where *a* and *b* are real numbers and *i* is the imaginary unit $\sqrt{-1}$. Complex numbers attach meaning to and allow calculations with the square root of negative numbers. When graphing complex numbers, we plot the real number *a* on the x-axis (labeled R for real) and *b* on the y-axis (labeled I for imaginary). The modulus is the length of the **vector** from the origin to the position of the complex number on the graph. The argument is the angle the vector makes with the horizontal axis R.



Argument = θ
Modulus = $\sqrt{(3^2)+(2^2)} = \sqrt{13}$

When graphing complex numbers in vector form, it is often desirable to present the numbers as **ordered pairs**. Thus, in the above example, the ordered pair of the plotted point is (3,2) when graphed in the complex plane.

An alternative representation of a complex number is the **polar form**. We can represent any complex number z with the following equation.

$z = r(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument.

Thus, from the example above, the polar form of $3 + 2i$ is

$$\sqrt{13}(\cos(\tan^{-1}\frac{2}{3}) + i\sin(\tan^{-1}\frac{2}{3})) .$$

The final alternative representation of a complex number is the **exponential form**. We can represent any complex number z with the following equation:

$$z = re^{i\vartheta}$$

Thus, from the example above, the exponential form of $3 + 2i$ is

$$\sqrt{13}e^{0.588i}$$ (with the argument, θ, given in radians).

Writing complex numbers in polar form simplifies multiplication and division applications. Conversely, Cartesian notation ($a + bi$) makes addition and subtraction applications easier to manage. In addition, representing complex numbers as vectors enables addition and subtraction in graphical form. Finally, writing complex numbers in exponential form is often more convenient than polar form because it eliminates cumbersome trigonometric relations.