I. PROCESSES AND APPLICATIONS

01 Understand how to communicate mathematical content and concepts, and develop and utilize a variety of problem-solving techniques.

Successful math teachers introduce their students to multiple problem solving strategies and create a classroom environment where free thought and experimentation are encouraged. Teachers can promote problem solving by allowing multiple attempts at problems, giving credit for reworking test or homework problems and encouraging the sharing of ideas through class discussion. There are several specific problem solving skills with which teachers should be familiar.

The **guess-and-check** strategy calls for students to make an initial guess at the solution, check the answer, and use the outcome of to guide the next guess. With each successive guess, the student should get closer to the correct answer. Constructing a table from the guesses can help organize the data.

Example:

There are 100 coins in a jar. 10 are dimes. The rest are pennies and nickels. There are twice as many pennies as nickels. How many pennies and nickels are in the jar?

There are 90 total nickels and pennies in the jar (100 coins – 10 dimes).

There are twice as many pennies as nickels. Make guesses that fulfill the criteria and adjust based on the answer found. Continue until we find the correct answer, 60 pennies and 30 nickels.

Number of Pennies	Number of Nickels	Total Number of Pennies and Nickels
40	20	60
80	40	120
70	35	105
60	30	90

When solving a problem where the final result and the steps to reach the result are given, students must **work backwards** to determine what the starting point must have been.

Example:

John subtracted seven from his age, and divided the result by 3. The final result was 4. What is John's age?

> Work backward by reversing the operations.
> $4 \times 3 = 12$;
> $12 + 7 = 19$
> John is 19 years old.

02 **Understand how to apply appropriate reasoning techniques to concepts, procedures, and conjectures, and make connections with and among the various branches of mathematics and other disciplines.**

Estimation and testing for **reasonableness** are related skills students should employ prior to and after solving a problem. These skills are particularly important when students use calculators to find answers.

Example:

Find the sum of 4387 + 7226 + 5893.

4300 + 7200 + 5800 = 17300	Estimation.
4387 + 7226 + 5893 = 17506	Actual sum.

By comparing the estimate to the actual sum, students can determine that their answer is reasonable.

The **questioning technique** is a mathematic process skill in which students devise questions to clarify the problem, eliminate possible solutions, and simplify the problem solving process. By developing and attempting to answer simple questions, students can tackle difficult and complex problems.

Observation-inference is another mathematic process skill that used regularly in statistics. We can use the data gathered or observed from a sample of the population to make inferences about traits and qualities of the population as a whole. For example, if we observe that 40% of voters in our sample favors Candidate A, then we can infer that 40% of the entire voting population favors Candidate A. Successful use of observation-inference depends on accurate observation and representative sampling.

03 Understand how to select, integrate, and use appropriate technologies

The use of supplementary materials in the classroom can greatly enhance the learning experience by stimulating student interest and satisfying different learning styles. Manipulatives, models, and technology are examples of tools available to teachers.

Manipulatives are materials that students can physically handle and move. Manipulatives allow students to understand mathematic concepts by allowing them to see concrete examples of abstract processes. Manipulatives are attractive to students because they appeal to the students' visual and tactile senses. Available for all levels of math, manipulatives are useful tools for reinforcing operations and concepts. They are not, however, a substitute for the development of sound computational skills.

Models are another means of representing mathematical concepts by relating the concepts to real-world situations. Teachers must choose wisely when devising and selecting models because, to be effective, models must be applied properly. For example, a building with floors above and below ground is a good model for introducing the concept of negative numbers. It would be difficult, however, to use the buildling model in teaching subtraction of negative numbers.

Finally, there are many forms of **technology** available to math teachers. For example, students can test their understanding of math concepts by working on skill specific computer programs and websites. Graphing calculators can help students visualize the graphs of functions. Teachers can also enhance their lectures and classroom presentations by creating multimedia presentations.

II. NUMBER SENSE AND MEASUREMENT

04 Understand the concepts of number, number theory, and numeration systems.

Prime numbers are numbers that can only be factored into 1 and the number itself. When factoring into prime factors, all the factors must be numbers that cannot be factored again (without using 1). Initially numbers can be factored into any 2 factors. Check each resulting factor to see if it can be factored again. Continue factoring until all remaining factors are prime. This is the list of prime factors. Regardless of what way the original number was factored, the final list of prime factors will always be the same.

Example: Factor 30 into prime factors.

Factor 30 into any 2 factors.
$5 \cdot 6$ Now factor the 6.
$5 \cdot 2 \cdot 3$ These are all prime factors.

Factor 30 into any 2 factors.
$3 \cdot 10$ Now factor the 10.
$3 \cdot 2 \cdot 5$ These are the same prime factors even though the original factors were different.

Example: Factor 240 into prime factors.

Factor 240 into any 2 factors.
$24 \cdot 10$ Now factor both 24 and 10.
$4 \cdot 6 \cdot 2 \cdot 5$ Now factor both 4 and 6.
$2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 5$ These are prime factors.

This can also be written as $2^4 \cdot 3 \cdot 5$.

a. A number is divisible by 2 if that number is an even number (which means it ends in 0,2,4,6 or 8).

> 1,354 ends in 4, so it is divisible by 2. 240,685 ends in a 5, so it is not divisible by 2.

b. A number is divisible by 3 if the sum of its digits is evenly divisible by 3.

The sum of the digits of 964 is 9+6+4 = 19. Since 19 is not divisible by 3, neither is 964. The digits of 86,514 is 8+6+5+1+4 = 24. Since 24 is divisible by 3, 86,514 is also divisible by 3.

c. A number is divisible by 4 if the number in its last 2 digits is evenly divisible by 4.

The number 113,336 ends with the number 36 in the last 2 columns. Since 36 is divisible by 4, then 113,336 is also divisible by 4.

The number 135,627 ends with the number 27 in the last 2 columns. Since 27 is not evenly divisible by 4, then 135,627 is also not divisible by 4.

d. A number is divisible by 5 if the number ends in either a 5 or a 0.

225 ends with a 5 so it is divisible by 5. The number 470 is also divisible by 5 because its last digit is a 0. 2,358 is not divisible by 5 because its last digit is an 8, not a 5 or a 0.

e. A number is divisible by 6 if the number is even and the sum of its digits is evenly divisible by 3.

4,950 is an even number and its digits add to 18. (4+9+5+0 = 18) Since the number is even and the sum of its digits is 18 (which is divisible by 3), then 4950 is divisible by 6. 326 is an even number, but its digits add up to 11. Since 11 is not divisible by 3, then 326 is not divisible by 6. 698,135 is not an even number, so it cannot possibly be divided evenly by 6.

f. A number is divisible by 8 if the number in its last 3 digits is evenly divisible by 8.

The number 113,336 ends with the 3-digit number 336 in the last 3 places. Since 336 is divisible by 8, then 113,336 is also divisible by 8.
The number 465,627 ends with the number 627 in the last 3 places. Since 627 is not evenly divisible by 8, then 465,627 is also not divisible by 8.

g. A number is divisible by 9 if the sum of its digits is evenly divisible by 9.

The sum of the digits of 874 is 8+7+4 = 19. Since 19 is not divisible by 9, neither is 874. The digits of 116,514 is 1+1+6+5+1+4 = 18. Since 18 is divisible by 9, 116,514 is also divisible by 9.

h. A number is divisible by 10 if the number ends in the digit 0.

305 ends with a 5 so it is not divisible by 10. The number 2,030,270 is divisible by 10 because its last digit is a 0. 42,978 is not divisible by 10 because its last digit is an 8, not a 0.

i. Why these rules work.

All even numbers are divisible by 2 by definition. A 2-digit number (with T as the tens digit and U as the ones digit) has as its sum of the digits,
T + U. Suppose this sum of T + U is divisible by 3. Then it equals 3 times some constant, K. So, T + U = 3K. Solving this for U, U = 3K - T. The original 2 digit number would be represented by 10T + U. Substituting
3K - T in place of U, this 2-digit number becomes
10T + U = 10T + (3K - T) = 9T + 3K. This 2-digit number is clearly divisible by 3, since each term is divisible by 3. Therefore, if the sum of the digits of a number is divisible by 3, then the number itself is also divisible by 3. Since 4 divides evenly into 100, 200, or 300, 4 will divide evenly into any amount of hundreds. The only part of a number that determines if 4 will divide into it evenly is the number in the last 2 places. Numbers divisible by 5 end in 5 or 0. This is clear if you look at the answers to the multiplication table for 5. Answers to the multiplication table for 6 are all even numbers. Since 6 factors into 2 times 3, the divisibility rules for 2 and 3 must both work. Any number of thousands is divisible by 8. Only the last 3 places of the number determine whether or not it is divisible by 8.

A 2 digit number (with T as the tens digit and U as the ones digit) has as its sum of the digits, T + U. Suppose this sum of T + U is divisible by 9. Then it equals 9 times some constant, K. So, T + U = 9K. Solving this for U,

U = 9K - T. The original 2-digit number would be represented by 10T + U. Substituting 9K - T in place of U, this 2-digit number becomes

10T + U = 10T + (9K - T) = 9T + 9K. This 2-digit number is clearly divisible by 9, since each term is divisible by 9. Therefore, if the sum of the digits of a number is divisible by 9, then the number itself is also divisible by 9. Numbers divisible by 10 must be multiples of 10 which all end in a zero.

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Prime numbers are whole numbers greater than 1 that have only 2 factors, 1 and the number itself. Examples of prime numbers are 2,3,5,7,11,13,17, or 19. Note that 2 is the only even prime number.

Composite numbers are whole numbers that have more than 2 different factors. For example 9 is composite because besides factors of 1 and 9, 3 is also a factor. 70 is also composite because besides the factors of 1 and 70, the numbers 2,5,7,10,14, and 35 are also all factors.

Remember that the number 1 is neither prime nor composite.

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The **exponent form** is a shortcut method to write repeated multiplication. The **base** is the factor. The **exponent** tells how many times that number is multiplied by itself.

The following are basic rules for exponents:

- $a^1 = a$ for all values of a; thus $17^1 = 17$
- $b^0 = 1$ for all values of b; thus $24^0 = 1$
- $10^n = 1$ with *n* zeros; thus $10^6 = 1{,}000{,}000$

05 **Understand properties of the real and complex number systems as they apply to algorithms of operations.**

To convert a fraction to a decimal, simply divide the numerator (top) by the denominator (bottom). Use long division if necessary.

If a decimal has a fixed number of digits, the decimal is said to be terminating. To write such a decimal as a fraction, first determine what place value the farthest right digit is in, for example: tenths, hundredths, thousandths, ten thousandths, hundred thousands, etc. Then drop the decimal and place the string of digits over the number given by the place value.

If a decimal continues forever by repeating a string of digits, the decimal is said to be repeating. To write a repeating decimal as a fraction, follow these steps.

a. Let $x =$ the repeating decimal
(ex. $x = .716716716...$)
b. Multiply x by the multiple of ten that will move the decimal just to the right of the repeating block of digits.
(ex. $1000x = 716.716716...$)
c. Subtract the first equation from the second.
(ex. $1000x - x = 716.716.716... - .716716...$)
d. Simplify and solve this equation. The repeating block of digits will subtract out.
(ex. $999x = 716$ so $x = 716/999$)
e. The solution will be the fraction for the repeating decimal.

The real number properties are best explained in terms of a small set of numbers. For each property, a given set will be provided.

Axioms of Addition

Closure—For all real numbers *a* and *b*, $a + b$ is a unique real number.

Associative—For all real numbers *a, b,* and *c,* $(a + b) + c = a + (b + c)$.

Additive Identity—There exists a unique real number 0 (zero) such that $a + 0 = 0 + a = a$ for every real number *a.*

Additive Inverses—For each real number *a*, there exists a real number $–a$ (the opposite of *a*) such that $a + (-a) = (-a) + a = 0$.

Commutative—For all real numbers *a* and *b,* $a + b = b + a$.

Axioms of Multiplication

Closure—For all real numbers *a* and *b, ab* is a unique real number.

Associative—For all real numbers *a, b,* and *c,* $(ab)c = a(bc)$.

Multiplicative Identity—There exists a unique nonzero real number 1 (one) such that $1 \cdot a = a \cdot 1 = a$.

Multiplicative Inverses—For each nonzero rel number, there exists a real number $1/a$ (the reciprocal of a) such that $a(1/a) = (1/a)a = 1$.

Commutative—For all real numbers *a* and *b*, $ab = ba$.

The Distributive Axiom of Multiplication over Addition

For all real numbers *a, b,* and *c,* $a(b + c) = ab + ac$.

a. **Natural numbers**--the counting numbers, $1, 2, 3, \ldots$

b. **Whole numbers**--the counting numbers along with zero, $0, 1, 2 \ldots$

c. **Integers**--the counting numbers, their opposites, and zero, $\ldots, {}^{-}1, 0, 1, \ldots$

d. **Rationals**--all of the fractions that can be formed from the whole numbers. Zero cannot be the denominator. In decimal form, these numbers will either be terminating or repeating decimals. Simplify square roots to determine if the number can be written as a fraction.

e. **Irrationals**--real numbers that cannot be written as a fraction. The decimal forms of these numbers are neither terminating nor repeating. <u>Examples:</u> $\pi, e, \sqrt{2}$, etc.

f. **Real numbers**--the set of numbers obtained by combining the rationals and irrationals. Complex numbers, i.e. numbers that involve i or $\sqrt{{}^{-}1}$, are not real numbers.

* * *

The **Denseness Property** of real numbers states that, if all real numbers are ordered from least to greatest on a number line, there is an infinite set of real numbers between any two given numbers on the line.

<u>Example</u>:

Between 7.6 and 7.7, there is the rational number 7.65 in the set of real numbers.
Between 3 and 4 there exists no other natural number.

For standardization purposes, there is an accepted order in which operations are performed in any given algebraic expression. The following pneumonic is often used for the order in which operations are performed.

Please	Parentheses	
Excuse	Exponents	
My	Multiply	Multiply or Divide depending on which
Dear	Divide	operation is encountered first from left to right.
Aunt	Add	Add or Subtract depending on which
Sally	Subtract	operation is encountered first from left to right.

*   *   *

Subtraction is the inverse of Addition, and vice-versa.
Division is the inverse of Multiplication, and vice-versa.
Taking a square root is the inverse of squaring, and vice-versa.

These inverse operations are used when solving equations.

In order to add or subtract rational expressions, they must have a common denominator. If they don't have a common denominator, then factor the denominators to determine what factors are missing from each denominator to make the LCD. Multiply both numerator and denominator by the missing factor(s). Once the fractions have a common denominator, add or subtract their numerators, but keep the common denominator the same. Factor the numerator if possible and reduce if there are any factors that can be cancelled.

1. Find the least common denominator for $6a^3b^2$ and $4ab^3$.

These factor into $2 \cdot 3 \cdot a^3 \cdot b^2$ and $2 \cdot 2 \cdot a \cdot b^3$.
The first expression must be multiplied by another 2 and b.
The other expression must be multiplied by 3 and a^2.
Then both expressions would be
$2 \cdot 2 \cdot 3 \cdot a^3 \cdot b^3 = 12a^3b^3 =$ LCD.

2. Find the LCD for $x^2 - 4$, $x^2 + 5x + 6$, and $x^2 + x - 6$.

$x^2 - 4$ factors into $(x-2)(x+2)$
$x^2 + 5x + 6$ factors into $(x+3)(x+2)$
$x^2 + x - 6$ factors into $(x+3)(x-2)$

To make these lists of factors the same, they must all be $(x+3)(x+2)(x-2)$. This is the LCD.

3.

$$\frac{5}{6a^3b^2} + \frac{1}{4ab^3} = \frac{5(2b)}{6a^3b^2(2b)} + \frac{1(3a^2)}{4ab^3(3a^2)} = \frac{10a}{12a^3b^3} + \frac{3b^2}{12a^3b^3} = \frac{10a + 3b^2}{12a^3b^3}$$

This will not reduce as all 3 terms are not divisible by anything.

4.

$$\frac{2}{x^2-4}-\frac{3}{x^2+5x+6}+\frac{7}{x^2+x-6}=$$

$$\frac{2}{(x-2)(x+2)}-\frac{3}{(x+3)(x+2)}+\frac{7}{(x+3)(x-2)}=$$

$$\frac{2(x+3)}{(x-2)(x+2)(x+3)}-\frac{3(x-2)}{(x+3)(x+2)(x-2)}+\frac{7(x+2)}{(x+3)(x-2)(x+2)}=$$

$$\frac{2x+6}{(x-2)(x+2)(x+3)}-\frac{3x-6}{(x+3)(x+2)(x-2)}+\frac{7x+14}{(x+3)(x-2)(x+2)}=$$

$$\frac{2x+6-(3x-6)+7x+14}{(x+3)(x-2)(x+2)}=\frac{6x+26}{(x+3)(x-2)(x+2)}$$

This will not reduce.

Try These:

5. $\frac{6}{x-3}+\frac{2}{x+7}$

6. $\frac{5}{4a^2b^5}+\frac{3}{5a^4b^3}$

7. $\frac{x+3}{x^2-25}+\frac{x-6}{x^2-2x-15}$