SUBAREA I MATHEMATICAL REASONING, PROBLEM SOLVING, AND METHODS, NUMBER CONCEPTS, AND THE HISTORICAL DEVELOPMENT OF MATHEMATICS

0001 Understand and apply problem-solving strategies, connections among different mathematical ideas, and mathematical modeling to solve application problems encountered in life.

Successful math teachers introduce their students to multiple problem solving strategies and create a classroom environment where free thought and experimentation are encouraged. Teachers can promote problem solving by allowing multiple attempts at problems, giving credit for reworking test or homework problems, and encouraging the sharing of ideas through class discussion. There are several specific problem solving skills with which teachers should be familiar.

The **guess-and-check** strategy calls for students to make an initial guess at the solution, check the answer, and use the outcome of to guide the next guess. With each successive guess, the student should get closer to the correct answer. Constructing a table from the guesses can help organize the data.

Example:

There are 100 coins in a jar. 10 are dimes. The rest are pennies and nickels. There are twice as many pennies as nickels. How many pennies and nickels are in the jar?

There are 90 total nickels and pennies in the jar (100 coins – 10 dimes).

There are twice as many pennies as nickels. Make guesses that fulfill the criteria and adjust based on the answer found. Continue until we find the correct answer, 60 pennies and 30 nickels.

Number of Pennies	Number of Nickels	Total Number of Pennies and Nickels
40	20	60
80	40	120
70	35	105
60	30	90

When solving a problem where the final result and the steps to reach the result are given, students must **work backwards** to determine what the starting point must have been.

Example:

John subtracted seven from his age, and divided the result by 3. The final result was 4. What is John's age?

> Work backward by reversing the operations.
> 4 x 3 = 12;
> 12 + 7 = 19
> John is 19 years old.

Estimation and testing for **reasonableness** are related skills students should employ prior to and after solving a problem. These skills are particularly important when students use calculators to find answers.

Example:

Find the sum of 4387 + 7226 + 5893.

4300 + 7200 + 5800 = 17300	Estimation.
4387 + 7226 + 5893 = 17506	Actual sum.

By comparing the estimate to the actual sum, students can determine that their answer is reasonable.

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Recognition and understanding of the relationships between concepts and topics is of great value in mathematical problem solving and the explanation of more complex processes.

For instance, multiplication is simply repeated addition. This relationship explains the concept of variable addition. We can show that the expression 4x + 3x = 7x is true by rewriting 4 times x and 3 times x as repeated addition, yielding the expression (x + x + x + x) + (x + x + x). Thus, because of the relationship between multiplication and addition, variable addition is accomplished by coefficient addition.

Another example of a mathematical relationship is powers as repeated multiplication. This relationship explains the rules of exponent operations. For instance, the multiplication of exponential terms with like bases is accomplished by the addition of the exponents.

$$2^2 \times 2^5 = 2^7$$

$$(2 \times 2) + (2 \times 2 \times 2 \times 2 \times 2) = 2^{2+5} = 2^7$$

This rule yields the general formula for the product of exponential terms, $z^m \times z^n = z^{m+n}$, that is useful in problem solving.

Because mathematics problems and concepts are often presented in written form students must have the ability to interpret written presentations and reproduce the concepts in symbolic form to facilitate manipulation and problem solving. Correct interpretation requires a sound understanding of the vocabulary of mathematics.

There are many types of written presentations of mathematics and the following are but two examples.

Examples:

1. The square of the hypotenuse of a right triangle is equivalent to the sum of the squares of the two legs.

$$a^2 + b^2 = c^2$$

2. Find the velocity of an object at time *t* given the objects position function is $f(t) = t^2 – 8t + 9$.

   The velocity at a given time (t) is equal to the value of the derivative of the position function at *t*. Thus…

   $$v(t) = f'(t) = 2t – 8$$

   The velocity after *t* seconds is $2t – 8$.

0002 Understand principles of mathematical reasoning and techniques for communicating mathematical ideas.

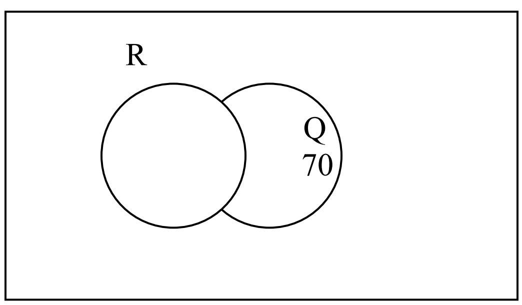
Examples, illustrations, and symbolic representations are useful tools in explaining and understanding mathematical concepts. The ability to create examples and alternative methods of expression allows students to solve real world problems and better communicate their thoughts.

Concrete examples are real world applications of mathematical concepts. For example, measuring the shadow produced by a tree or building is a real world application of trigonometric functions, acceleration or velocity of a car is an application of derivatives, and finding the volume or area of a swimming pool is a real world application of geometric principles.

Pictorial illustrations of mathematic concepts help clarify difficult ideas and simplify problem solving.

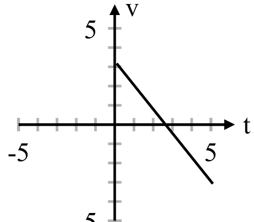
Examples:

1. Rectangle R represents the 300 students in School A. Circle P represents the 150 students that participated in band. Circle Q represents the 170 students that participated in a sport. 70 students participated in both band and a sport.



Pictorial representation of above situation.

2. A ball rolls up an incline and rolls back to its original position. Create a graph of the velocity of the ball.



Velocity starts out at its maximum as the ball begins to roll, decreases to zero at the top of the incline, and returns to the maximum in the opposite direction at the bottom of the incline.

Symbolic representation is the basic language of mathematics. Converting data to symbols allows for easy manipulation and problem solving. Students should have the ability to recognize what the symbolic notation represents and convert information into symbolic form. For example, from the graph of a line, students should have the ability to determine the slope and intercepts and derive the line's equation from the observed data. Another possible application of symbolic representation is the formulation of algebraic expressions and relations from data presented in word problem form.

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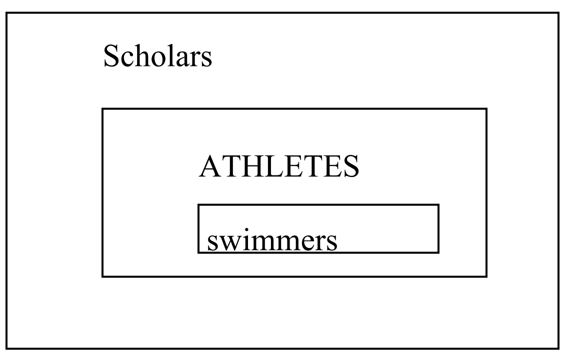
Conditional statements can be diagrammed using a **Venn diagram**. A diagram can be drawn with one figure inside another figure. The inner figure represents the hypothesis. The outer figure represents the conclusion. If the hypothesis is taken to be true, then you are located inside the inner figure. If you are located in the inner figure then you are also inside the outer figure, so that proves the conclusion is true. Sometimes that conclusion can then be used as the hypothesis for another conditional, which can result in a second conclusion.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All athletes are scholars.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are an athlete, then you are a scholar.

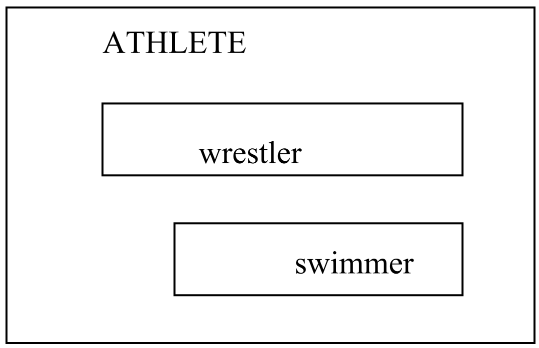


Clearly, if you are a swimmer, then you are also an athlete. This includes you in the group of scholars.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All wrestlers are athletes.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are a wrestler, then you are an athlete.



Clearly, if you are a swimmer or a wrestler, then you are also an athlete. This does NOT allow you to come to any other conclusions.

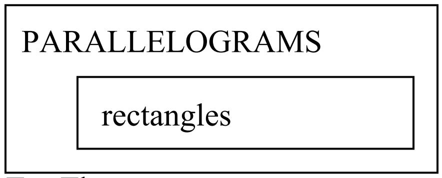
A swimmer may or may NOT also be a wrestler. Therefore, NO CONCLUSION IS POSSIBLE.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All rectangles are parallelograms.
Quadrilateral ABCD is not a parallelogram.

In "if-then" form, the first statement would be:
If a figure is a rectangle, then it is also a parallelogram.

Note that the second statement is the negation of the conclusion of statement one. Remember also that the contrapositive is logically equivalent to a given conditional. That is, **"If ⊔ q, then ⊔ p"**. Since" ABCD is NOT a parallelogram " is like saying **"If ⊔ q,"** then you can come to the conclusion **"then ⊔ p".** Therefore, the conclusion is ABCD is not a rectangle. Looking at the Venn diagram below, if all rectangles are parallelograms, then rectangles are included as part of the parallelograms. Since quadrilateral ABCD is not a parallelogram, that it is excluded from anywhere inside the parallelogram box. This allows you to conclude that ABCD can not be a rectangle either.



quadrilateral
ABCD

Try These:

What conclusion, if any, can be reached? Assume each statement is true, regardless of any personal beliefs.

1. If the Red Sox win the World Series, I will die.
   I died.

2. If an angle's measure is between 0° and 90°, then the angle is acute.
Angle B is not acute.

3. Students who do well in geometry will succeed in college.
   Annie is doing extremely well in geometry.

4. Left-handed people are witty and charming.
   You are left-handed.

* * *

Inductive thinking is the process of finding a pattern from a group of examples. That pattern is the conclusion that this set of examples seemed to indicate. It may be a correct conclusion or it may be an incorrect conclusion because other examples may not follow the predicted pattern.

Deductive thinking is the process of arriving at a conclusion based on other statements that are all known to be true, such as theorems, axiomspostulates, or postulates. Conclusions found by deductive thinking based on true statements will **always** be true.

Examples :

Suppose:
On Monday Mr.Peterson eats breakfast at McDonalds.
On Tuesday Mr.Peterson eats breakfast at McDonalds.
On Wednesday Mr.Peterson eats breakfast at McDonalds.
On Thursday Mr.Peterson eats breakfast at McDonalds again.

Conclusion: On Friday Mr. Peterson will eat breakfast at McDonalds again.

This is a conclusion based on inductive reasoning. Based on several days observations, you conclude that Mr. Peterson will eat at McDonalds. This may or may not be true, but it is a conclusion arrived at by inductive thinking.