SUBAREA I–MATHEMATIC REASONING AND COMMUNICATION

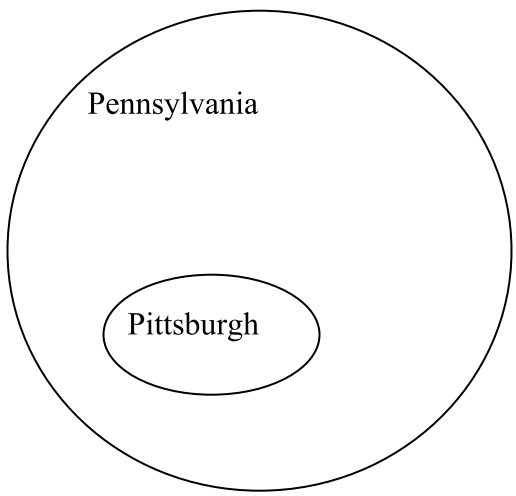
0001. Understand reasoning processes, including inductive and deductive logic and symbolic logic.

Conditional statements are frequently written in "**if-then**" form. The "if" clause of the conditional is known as the **hypothesis**, and the "then" clause is called the **conclusion**. In a proof, the hypothesis is the information that is assumed to be true, while the conclusion is what is to be proven true. A conditional is considered to be of the form:

If p, then q

p is the hypothesis. q is the conclusion.

Conditional statements can be diagrammed using a **Venn diagram**. A diagram can be drawn with one circle inside another circle. The inner circle represents the hypothesis. The outer circle represents the conclusion. If the hypothesis is taken to be true, then you are located inside the inner circle. If you are located in the inner circle then you are also inside the outer circle, so that proves the conclusion is true.



Example:
If an angle has a measure of 90 degrees, then it is a right angle.

In this statement "an angle has a measure of 90 degrees" is the hypothesis.

In this statement "it is a right angle" is the conclusion.

Example:

If you are in Pittsburgh, then you are in Pennsylvania.

In this statement "you are in Pittsburgh" is the hypothesis.

In this statement "you are in Pennsylvania" is the conclusion.

Conditional: If p, then q p is the hypothesis. q is the conclusion.

Inverse: If ⊔ p, then ⊔ q. Negate both the hypothesis (If not p, then not q) and the conclusion from the original conditional.

Converse : If q, then p. Reverse the 2 clauses. The original hypothesis becomes the conclusion. The original conclusion then becomes the new hypothesis.

Contrapositive: If ⊔ q, then ⊔ p. Reverse the 2 clauses. The If not q, then not p original hypothesis becomes the conclusion. The original conclusion then becomes the new hypothesis. THEN negate both the new hypothesis and the new conclusion.

<u>Example</u>: Given the **conditional**:

If an angle has 60°, then it is an acute angle.

Its **inverse**, in the form "If ⊔ p, then ⊔ q", would be:

If an angle doesn't have 60°, then it is not an acute angle.

NOTICE that the inverse is not true, even though the conditional statement was true.

Its **converse**, in the form "If q, then p", would be:

If an angle is an acute angle, then it has 60°.

NOTICE that the converse is not true, even though the conditional statement was true.

Its **contrapositive**, in the form "If   q, then   p", would be:

If an angle isn't an acute angle, then it doesn't have 60°.

NOTICE that the contrapositive is true, assuming
the original conditional statement was true.

TIP: If you are asked to pick a statement that is logically equivalent to a given conditional, look for the contra-positive. The inverse and converse are not always logically equivalent to every conditional. The contra-positive is ALWAYS logically equivalent.

Find the inverse, converse and contrapositive of the following conditional statement. Also determine if each of the 4 statements is true or false.

Conditional: If $x = 5$, then $x^2 - 25 = 0$.   TRUE
Inverse: If $x \neq 5$, then $x^2 - 25 \neq 0$.   FALSE, *x* could be $^{-}5$
Converse: If $x^2 - 25 = 0$, then $x = 5$.   FALSE, *x* could be $^{-}5$
Contrapositive: If $x^2 - 25 \neq 0$, then $x \neq 5$. TRUE

Conditional: If $x = 5$, then $6x = 30$.   TRUE
Inverse: If $x \neq 5$, then $6x \neq 30$.   TRUE
Converse: If $6x = 30$, then $x = 5$.   TRUE
Contrapositive: If $6x \neq 30$, then $x \neq 5$. TRUE

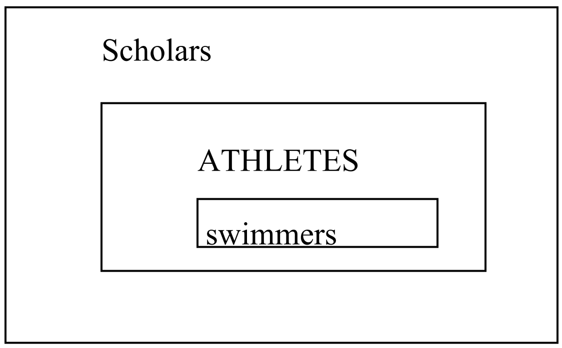
Sometimes, as in this example, all 4 statements can be logically equivalent; however, the only statement that will always be logically equivalent to the original conditional is the contrapositive.

Conditional statements can be diagrammed using a **Venn diagram**. A diagram can be drawn with one figure inside another figure. The inner figure represents the hypothesis. The outer figure represents the conclusion. If the hypothesis is taken to be true, then you are located inside the inner figure. If you are located in the inner figure then you are also inside the outer figure, so that proves the conclusion is true. Sometimes that conclusion can then be used as the hypothesis for another conditional, which can result in a second conclusion.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All athletes are scholars.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are an athlete, then you are a scholar.

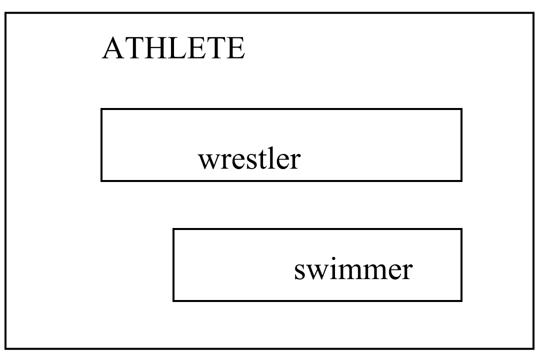


Clearly, if you are a swimmer, then you are also an athlete. This includes you in the group of scholars.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All wrestlers are athletes.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are a wrestler, then you are an athlete.



Clearly, if you are a swimmer or a wrestler, then you are also an athlete. This does NOT allow you to come to any other conclusions.

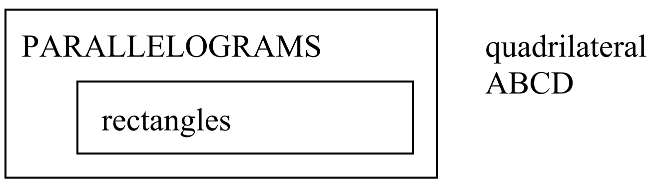
A swimmer may or may NOT also be a wrestler. Therefore, NO CONCLUSION IS POSSIBLE.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All rectangles are parallelograms.
Quadrilateral ABCD is not a parallelogram.

In "if-then" form, the first statement would be:
If a figure is a rectangle, then it is also a parallelogram.

Note that the second statement is the negation of the conclusion of statement one. Remember also that the contrapositive is logically equivalent to a given conditional. That is, **"If ⊔ q, then ⊔ p"**. Since" ABCD is NOT a parallelogram " is like saying **"If ⊔ q,"** then you can come to the conclusion **"then ⊔ p".** Therefore, the conclusion is ABCD is not a rectangle. Looking at the Venn diagram below, if all rectangles are parallelograms, then rectangles are included as part of the parallelograms. Since quadrilateral ABCD is not a parallelogram, that it is excluded from anywhere inside the parallelogram box. This allows you to conclude that ABCD can not be a rectangle either.



Try These:

What conclusion, if any, can be reached? Assume each statement is true, regardless of any personal beliefs.

1. If the Red Sox win the World Series, I will die.
   I died.

2. If an angle's measure is between 0° and 90°, then the angle is acute.
Angle B is not acute.

3. Students who do well in geometry will succeed in college.
   Annie is doing extremely well in geometry.

4. Left-handed people are witty and charming.
   You are left-handed.

The only **undefined terms** are point, line and plane.

Definitions are explanations of all mathematical terms except those that are undefined.

Postulates are mathematical statements that are accepted as true statements without providing a proof.

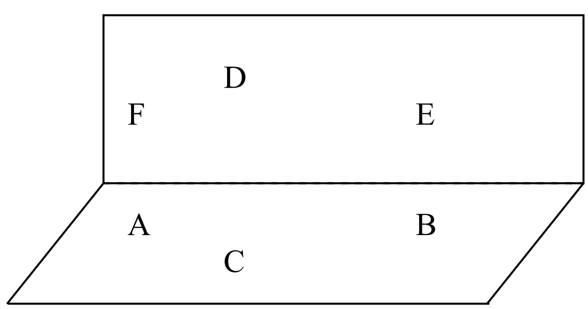
Theorems are mathematical statements that can be proven to be true based on postulates, definitions, algebraic properties, given information, and previously proved theorems.

The **3 undefined terms of geometry** are point, line, and plane.

A plane is a flat surface that extends forever in two dimensions. It has no ends or edges. It has no thickness to it. It is usually drawn as a parallelogram that can be named either by 3 non-collinear points (3 points that are not on the same line) on the plane or by placing a letter in the corner of the plane that is not used elsewhere in the diagram.

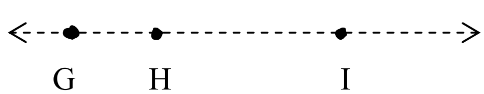
A line extends forever in one dimension. It is determined and named by 2 points that are on the line. The line consists of every point that is between those 2 points as well as the points that are on the "straight" extension each way. A line is drawn as a line segment with arrows facing opposite directions on each end to indicate that the line continues in both directions forever.

A point is a position in space, on a line, or on a plane. It has no thickness and no width. Only 1 line can go through any 2 points. A point is represented by a dot named by a single letter.



DEF is a plane.

ABC is another plane.

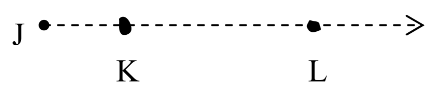


This line can be named by any two points on the line.

The line could be named $\overleftrightarrow{GH}$, $\overleftrightarrow{HI}$, $\overleftrightarrow{GI}$, $\overleftrightarrow{IG}$, $\overleftrightarrow{IH}$, or $\overleftrightarrow{HG}$. Any 2 points (letters) on the line can be used and their order is not important in naming a line.

In the previous diagrams, A, B, C, D, E, F, G, H, and I are all locations of individual points.

A ray is not an undefined term. A ray consists of all the points on a line starting at one given point and extending in only one of the two opposite directions along the line. The ray is named by naming 2 points on the ray. The first point must be the endpoint of the ray, while the second point can be any other point along the ray. The symbol for a ray is a ray above the 2 letters used to name it. The endpoint of the ray MUST be the first letter.



This ray could be named $\overrightarrow{JK}$ or $\overrightarrow{JL}$. It can not be called $\overrightarrow{KJ}$ or $\overrightarrow{LJ}$ or $\overrightarrow{LK}$ or $\overrightarrow{KL}$ because none of those names start with the endpoint, J.

The **distance** between 2 points on a number line is equal to the absolute value of the difference of the two numbers associated with the points.

If one point is located at "*a*" and the other point is at "*b*", then the distance between them is found by this formula:

$$\text{distance} = |a - b| \text{ or } |b - a|$$

If one point is located at $^{-}3$ and another point is located at 5, the distance between them is found by:

$$\text{distance} = |a - b| = |(^{-}3) - 5| = |^{-}8| = 8$$

In a **2 column proof**, the left side of the proof should be the given information, or statements that could be proved by deductive reasoning. The right column of the proof consists of the reasons used to determine that each statement to the left was verifiably true. The right side can identify given information, or state theorems, postulates, definitions or algebraic properties used to prove that particular line of the proof is true.

Assume the opposite of the conclusion. Keep your hypothesis given information the same. Proceed to develop the steps of the proof, looking for a statement that contradicts your original assumption or some other known fact. This contradiction indicates that the assumption you made at the beginning of the proof was incorrect; therefore, the original conclusion has to be true.

The following **algebraic postulates** are frequently used as reasons for statements in 2 column geometric properties:

Addition Property:

If $a = b$ and $c = d$, then $a + c = b + d$.

Subtraction Property:

If $a = b$ and $c = d$, then $a - c = b - d$.

Multiplication Property:

If $a = b$ and $c \neq 0$, then $ac = bc$.

Division Property:

If $a = b$ and $c \neq 0$, then $a/c = b/c$.

Reflexive Property:	$a = a$
Symmetric Property:	If $a = b$, then $b = a$.
Transitive Property:	If $a = b$ and $b = c$, then $a = c$.
Distributive Property:	$a(b + c) = ab + ac$
Substitution Property:	If $a = b$, then b may be substituted for a in any other expression (a may also be substituted for b).

Inductive thinking is the process of finding a pattern from a group of examples. That pattern is the conclusion that this set of examples seemed to indicate. It may be a correct conclusion or it may be an incorrect conclusion because other examples may not follow the predicted pattern.

Deductive thinking is the process of arriving at a conclusion based on other statements that are all known to be true, such as theorems, axiomspostulates, or postulates. Conclusions found by deductive thinking based on true statements will **always** be true.

Examples :

Suppose:
On Monday Mr.Peterson eats breakfast at McDonalds.
On Tuesday Mr.Peterson eats breakfast at McDonalds.
On Wednesday Mr.Peterson eats breakfast at McDonalds.
On Thursday Mr.Peterson eats breakfast at McDonalds again.

Conclusion: On Friday Mr. Peterson will eat breakfast at McDonalds again.

This is a conclusion based on inductive reasoning. Based on several days observations, you conclude that Mr. Peterson will eat at McDonalds. This may or may not be true, but it is a conclusion arrived at by inductive thinking.