DOMAIN I. NUMBER CONCEPTS

Competency 001 The teacher understands the structure of number systems, the development of a sense of quantity, and the relationship between quantity and symbolic representations.

A numeration system is a set of numbers represented by a set of symbols (numbers, letters, or pictographs). Numbers can be represented using different bases. Instead of the standard base 10 or decimal representation, a system may use any base set from 2 (binary) on up. The position of a number in a particular representation defines its exact value. Thus, the numeral 1 has a value of ten when represented in base 10 as "10". In base 2, the numeral 1 has a value of two when represented as "10". Early systems, such as the Babylonian, used position in relation to other numerals or column position for this purpose since they lacked a zero to represent an empty position.

A base of 2 uses only 0 and 1.

Decimal Binary Conversion		
Decimal	Binary	Place Value
1	1	2^0
2	10	2^1
4	100	2^2
8	1000	2^3

Thus, 9 in Base 10 becomes 1001 in Base 2.

9+4 = 13 (Base 10) becomes 1001 + 100 = 1101 (Base 2).

Fractions, ratios and other functions alter in the same way.

Computers use a base of 2 but combine it into 4 units called a byte to function in base 16 (hexadecimal). A base of 8 (octal) was also used by older computers.

The real number system includes all rational and irrational numbers.

Rational numbers, by definition, can be expressed as the ratio of two integers, $\frac{a}{b}$ where $b \neq 0$, for example $\frac{2}{3}$, $-\frac{4}{5}$, $5 = \frac{5}{1}$. The set of rational numbers includes integers, fractions and mixed numbers, terminating and repeating decimals. Every rational number can be expressed as a repeating or terminating decimal and can be shown on a number line.

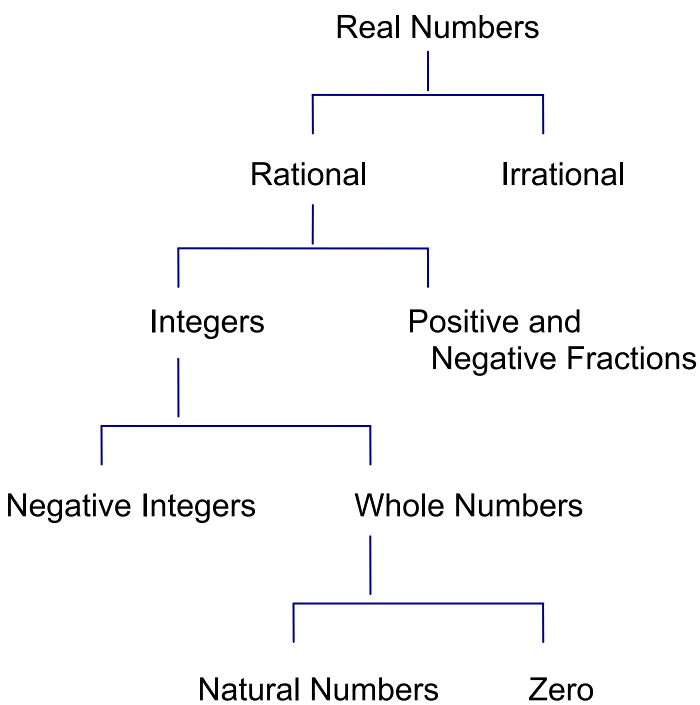
Integers are positive and negative whole numbers and zero.
...-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ...

Whole numbers are natural numbers and zero.
0, 1, 2, 3, ,4 ,5 ,6 ...

Natural numbers are the counting numbers.
1, 2, 3, 4, 5, 6, ...

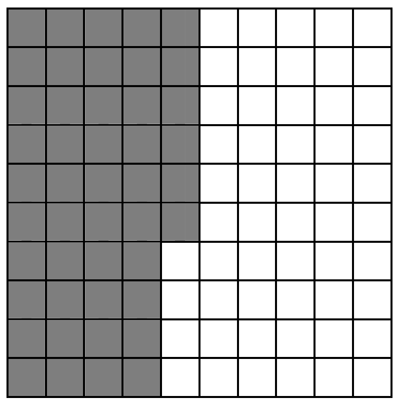
Irrational numbers are real numbers that cannot be written as the ratio of two integers. These are infinite non-repeating decimals.

<u>Examples</u>: $\sqrt{5} = 2.2360..$, pi = π = 3.1415927...

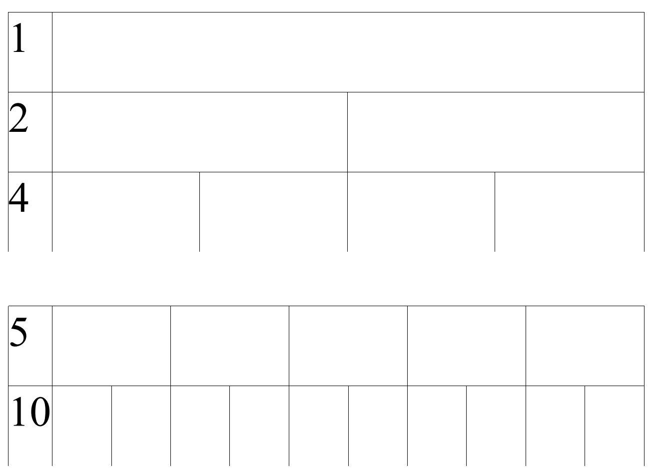


The following are examples of different ways in which numbers are represented:

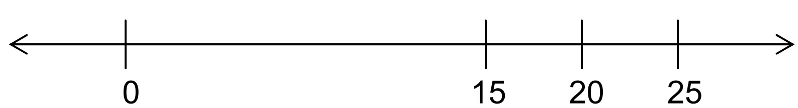
The shaded region below represents 47 out of 100 or 0.47 or $\frac{47}{100}$ or 47%.



Fraction Strips:



Number Lines:



Diagrams:

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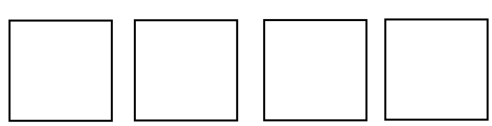

Percent means parts of one hundred. Numbers may be represented interchangeably as fractions, decimals or percents.

$$100\% = 1$$

In order to express a fraction as a decimal, convert it into an equivalent fraction whose denominator is a power of 10 (for example, 10, 100, 1000).

Examples:

$$\frac{1}{10} = 0.10 = 10\%$$

$$\frac{2}{5} = \frac{4}{10} = 0.40 = 40\%$$

$$\frac{1}{4} = \frac{25}{100} = 0.25 = 25\%$$

Alternatively, a fraction can be converted into a decimal by dividing the numerator by the denominator.

Example:

$$\frac{3}{8} = 8\overset{0.375}{\overline{)3.000}} = 37.5\%$$

Common Equivalents

$\frac{1}{2} = 0.5 = 50\%$	$\frac{2}{3} = 0.66\frac{2}{3} = 66\frac{2}{3}\%$
$\frac{1}{3} = 0.33\frac{1}{3} = 33\frac{1}{3}\%$	$\frac{5}{6} = 0.83\frac{1}{3} = 83\frac{1}{3}\%$
$\frac{1}{4} = 0.25 = 25\%$	$\frac{3}{8} = 0.37\frac{1}{2} = 37\frac{1}{2}\%$
$\frac{1}{5} = 0.2 = 20\%$	$\frac{5}{8} = 0.62\frac{1}{2} = 62\frac{1}{2}\%$
$\frac{1}{6} = 0.16\frac{2}{3} = 16\frac{2}{3}\%$	$\frac{7}{8} = 0.87\frac{1}{2} = 87\frac{1}{2}\%$
$\frac{1}{8} = 0.12 = 12\%$	$1 = 1.0 = 100\%$

$$\frac{1}{10} = 0.1 = 10\%$$

A **decimal** can be converted to a **percent** by multiplying by 100, or merely moving the decimal point two places to the right. A **percent** can be converted to a **decimal** by dividing by 100, or moving the decimal point two places to the left.

Example: Convert the following decimals into percents and vice versa.

0.375 = 37.5%
0.7 = 70%
0.04 = 4 %
3.15 = 315 %
84% = 0.84
3 % = 0.03
60% = 0.6
110% = 1.1
$\frac{1}{2}$ % = 0.5% = 0.005

A **percent** can be converted to a **fraction** by placing it over 100 and reducing to simplest terms.

Example: Convert the following percents into fractions.

$32\% = \frac{32}{100} = \frac{8}{25}$

$6\% = \frac{6}{100} = \frac{3}{50}$

$111\% = \frac{111}{100} = 1\frac{11}{10}$

The **exponent form** is a shortcut way of writing repeated multiplication. The **base** is the factor. The **exponent** tells how many times that number is multiplied by itself.

Example: 3^4 is $3 \times 3 \times 3 \times 3 = 81$
where 3 is the base and 4 is the exponent.

x^2 is read "x squared"
y^3 is read "y cubed"

a^1 = a for all values of a; thus $17^1 = 17$
b^0 = 1 for all values of b; thus $24^0 = 1$

When 10 is raised to any power, the exponent gives the number of zeroes in the product.

Example: $10^7 = 10{,}000{,}000$

Scientific notation is a more convenient method for writing very large and very small numbers. It employs two factors. The first factor is a number between -10 and 10. The second factor is a power of 10. This notation is a shorthand way to express large numbers (like the weight of 100 freight cars in kilograms) or small numbers (like the weight of an atom in grams).

Example: Write 46,368,000 in scientific notation.

1. Introduce a decimal point. 46,368,000 = 46,368,000.0

2. Move the decimal place to **left** until only one nonzero digit is in front of it, in this case between the 4 and 6.

3. Count the number of digits the decimal point moved, in this case 7. This is the n^{th} the power of ten and is **positive** because the decimal point moved **left**.

Therefore, $46{,}368{,}000 = 4.6368 \times 10^7$

Example: Write 0.00397 in scientific notation.

1. Decimal point is already in place.

2. Move the decimal point to the **right** until there is only one nonzero digit in front of it, in this case between the 3 and 9.

3. Count the number of digits the decimal point moved, in this case 3. This is the n^{th} the power of ten and is **negative** because the decimal point moved **right**.

. Therefore, $0.00397 = 3.97 \times 10^{-3}$.

Example: Write 2.19×10^8 in standard form.

Since the exponent is positive, move the decimal point 8 places to the right, and add additional zeroes as needed.

$$2.19 \times 10^8 = 219{,}000{,}000$$

Example: Write 8.04×10^{-4} in standard form.

Move the decimal point 4 places to the left, writing additional zeroes as needed.

$$8.04 \times 10^{-4} = 0.000804$$

** Note: The first factor **must** be between -10 and 10.

Real numbers exhibit the following addition and multiplication properties where *a, b*, and *c* are real numbers.

Note: Multiplication is implied when there is no symbol between two variables. Thus, $a \times b$ can be written ab. Multiplication can also be indicated by a raised dot $\bullet$.

Closure
The sum or product of two real numbers is a real number.

$a + b$ is a real number.
Example: Since 2 and 5 are both real numbers, 7 is also a real number.

ab is a real number.
Example: Since 3 and 4 are both real numbers, 12 is also a real number.

Commutative property
The order of the addends or factors does not affect the sum or product.

$a + b = b + a$
Example: $5 + {}^{-}8 = {}^{-}8 + 5 = {}^{-}3$

$ab = ba$
Example: ${}^{-}2 \times 6 = 6 \times {}^{-}2 = {}^{-}12$

Associative property
The grouping of the addends or factors does not affect the sum or product.

$(a + b) + c = a + (b + c)$
Example: $({}^{-}2 + 7) + 5 = {}^{-}2 + (7 + 5)$
$5 + 5 = {}^{-}2 + 12 = 10$

$(ab)\,c = a\,(bc)$
Example: $(3 \times {}^{-}7) \times 5 = 3 \times ({}^{-}7 \times 5)$
${}^{-}21 \times 5 = 3 \times {}^{-}35 = {}^{-}105$

Distributive property
To multiply a sum by a number, multiply each addend by the number, then add the products.

$a\,(b + c) = ab + ac$
Example: $6 \times (-4 + 9) = (6 \times -4) + (6 \times 9)$
$6 \times 5 = -24 + 54 = 30$

Additive Identity (Property of Zero)
The sum of any number and zero is that number.

$a + 0 = a$
<u>Example</u>: 17 + 0 = 17

Multiplicative Identity (Property of One)
The product of any number and one is that number.

$a \cdot 1 = a$
<u>Example</u>: $^{-}34 \times 1 = {}^{-}34$

Additive Inverse (Property of Opposites)
The sum of any number and its negative is zero.

$a + -a = 0$
<u>Example</u>: 25 + –25 = 0

Multiplicative Inverse (Property of Reciprocals)
The product of any number and its reciprocal is one.

$a \times \frac{1}{a} = 1$

<u>Example</u>: $5 \times \frac{1}{5} = 1$

Property of denseness
Between any pair of rational numbers, there is at least one rational number. The set of natural numbers is <u>not</u> dense because between two consecutive natural numbers there may not exist another natural number.

<u>Example:</u>
Between 7.6 and 7.7, there is the rational number 7.65 in the set of real numbers. Between 3 and 4 there exists no other natural number.

PROPERTIES SATISFIED BY SUBSETS OF REAL NUMBERS

+	Closure	Commutative	Associative	Distributive	Identity	Inverse
Real	yes	yes	yes	yes	yes	yes
Rational	yes	yes	yes	yes	yes	yes
Irrational	yes	yes	yes	yes	no	yes
Integers	yes	yes	yes	yes	yes	yes
Fractions	yes	yes	yes	yes	yes	yes
Whole	yes	yes	yes	yes	yes	no
Natural	yes	yes	yes	yes	no	no
-	Closure	Commutative	Associative	Distributive	Identity	Inverse
Real	yes	no	no	no	yes	yes
Rational	yes	no	no	no	yes	yes
Irrational	no	no	no	no	no	no
Integers	yes	no	no	no	yes	yes
Fractions	yes	no	no	no	yes	yes
Whole	no	no	no	no	yes	no
Natural	no	no	no	no	no	no
×	Closure	Commutative	Associative	Distributive	Identity	Inverse
Real	yes	yes	yes	yes	yes	yes
Rational	yes	yes	yes	yes	yes	yes
Irrational	yes	yes	yes	yes	no	no
Integers	yes	yes	yes	yes	yes	no
Fractions	yes	yes	yes	yes	yes	yes
Whole	yes	yes	yes	yes	yes	no
Natural	yes	yes	yes	yes	yes	no
÷	Closure	Commutative	Associative	Distributive	Identity	Inverse
Real	yes	no	no	no	yes	yes
Rational	yes	no	no	no	yes	yes
Irrational	no	no	no	no	no	no
Integers	no	no	no	no	yes	no
Fractions	yes	no	no	no	yes	yes
Whole	no	no	no	no	yes	no
Natural	no	no	no	no	yes	no

Complex numbers can be written in the form *a* + *b*i where i represents $\sqrt{-1}$ and *a* and *b* are real numbers. *a* is the real part of the complex number and *b* is the imaginary part. If *b* = 0, then the number has no imaginary part and it is a real number. If $b \neq 0$ the number is imaginary.

Complex numbers are found when trying to solve equations with negative square roots.

Example: If $x^2 + 9 = 0$
then $x^2 = -9$
and $x = \sqrt{-9}$ or +3i and -3i

When dividing one complex number by another, you must eliminate the complex number in the denominator. If the complex number in the denominator is of the form $b\,\mathbf{i}$, multiply both the numerator and denominator by **i**. Remember to replace $\mathbf{i}^2$ with $-\mathbf{1}$ and then continue simplifying the fraction.

Example: Simplify $\frac{2+3i}{5i}$

Multiply by $\frac{i}{i}$

$$\frac{2+3i}{5i} \times \frac{i}{i} = \frac{(2+3i)\,i}{5i \cdot i} = \frac{2i+3i^2}{5i^2} = \frac{2i+3(-1)}{-5} = \frac{-3+2i}{-5} = \frac{3-2i}{5}$$

If the complex number in the denominator is of the form $a + b\,\mathbf{i}$, multiply both the numerator and denominator by **the conjugate of the denominator**. (The conjugate of a complex number is the number with the sign of its imaginary part reversed. The conjugate of $2 - 3i$ is $2 + 3i$. The conjugate of $^{-}6 + 11i$ is $^{-}6 - 11i$.) Multiply together the factors on the top and bottom of the fraction. Remember to replace $\mathbf{i}^2$ with $-\mathbf{1}$, combine like terms, and then continue simplifying the fraction.

Example: Simplify $\frac{4+7i}{6-5i}$

$\frac{4+7i}{6-5i}$ Multiply numerator and denominator by 6+5i , the conjugate of the denominator.

$$\frac{(4+7i)}{(6-5i)} \times \frac{(6+5i)}{(6+5i)} = \frac{24+20i+42i+35i^2}{36+30i-30i-25i^2} = \frac{24+62i+35(^{-}1)}{36-25(^{-}1)} = \frac{^{-}11+62i}{61}$$