# COMPETENCY 1
## KNOWLEDGE OF ALGEBRA

### SKILL 1.1 Apply the properties of real numbers: closure, commutative, associative, distributive, transitive, identities, and inverses

The properties of real numbers are best explained in terms of a small set of numbers. For each property, a sample set will be provided.

### Axioms of Addition

**Closure:** For all real numbers $a$ and $b$, $a + b$ is a unique real number.

**Associative:** For all real numbers $a$, $b$, and $c$, $(a + b) + c = a + (b + c)$.

**Additive Identity:** There exists a unique real number 0 (zero) such that $a + 0 = 0 + a = a$ for every real number $a$.

**Additive Inverses**: For each real number $a$, there exists a real number $^{-}a$ (the opposite of $a$) such that $a + (^{-}a) = (^{-}a) + a = 0$.

**Commutative:** For all real numbers $a$ and $b$, $a + b = b + a$.

### Axioms of Multiplication

**Closure:** For all real numbers $a$ and $b$, $ab$ is a unique real number.

**Associative:** For all real numbers $a$, $b$, and $c$, $(ab)c = a(bc)$.

**Multiplicative Identity:** There exists a unique nonzero real number 1 (one) such that $1 \cdot a = a \cdot 1$.

**Multiplicative Inverses:** For each nonzero real number, there exists a real number $\frac{1}{a}$ (the reciprocal of $a$) such that $a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1$.

**Commutative:** For all real numbers $a$ and $b$, $ab = ba$.

### Axiom of Multiplication and Addition

**Distributive Axiom of Multiplication over Addition:** For all real numbers $a$, $b$, and $c$, $a(b + c) = ab + ac$.

## SKILL 1.2 Solve linear equations and inequalities in one or two variables, symbolically or graphically

To solve an **equation** or **inequality** symbolically, follow these steps:

1. If there are parentheses, use the distributive property to eliminate them.
2. If there are fractions, determine their LCD (least common denominator). Multiply every term of the equation by the LCD. This will eliminate all of the fractions in the equation or inequality.
3. If there are decimals, find the largest decimal. Multiply each term of the equation or inequality by a power of 10 (10, 100, 1,000, etc.) with the same number of zeros as the largest decimal. This will eliminate all decimals in the equation or inequality.
4. Combine like terms on each side of the equation or inequality.
5. If there are variable terms on both sides of the equation or inequality, add or subtract one of those variable terms to move it to the other side. Combine like terms.
6. If there are constants on both sides of the equation or inequality, add or subtract one of those constants to move it to the other side. Combine like terms.
7. If there is a coefficient in front of the variable, divide both sides of the equation or inequality by this number. This will give the answer to an equation. For an inequality, however, remember: Dividing or multiplying an inequality by a negative number will reverse the direction of the inequality sign.

*Dividing or multiplying an inequality by a negative number will reverse the direction of the inequality sign.*

*Example: Solve $3(2x + 5) - 4x = 5(x + 9)$.*

In solving the equation symbolically, refer to the steps listed above.

| | |
|---|---|
| $6x + 15 - 4x = 5x + 45$ | ref. step 1 |
| $2x + 15 = 5x + 45$ | ref. step 4 |
| $-3x + 15 = 45$ | ref. step 5 |
| $-3x = 30$ | ref. step 6 |
| $x = -10$ | ref. step 7 |

*Example: Solve $\frac{1}{2}(5x + 34) = \frac{1}{4}(3x - 5)$.*

$\left(\frac{5}{2}\right)x + 17 = \left(\frac{3}{4}\right)x - \frac{5}{4}$ ref. step 1

The LCD of $\frac{5}{2}$, $\frac{3}{4}$, and $\frac{5}{4}$ is 4.

Multiply by the LCD of 4.

$$4\left[\left(\frac{5}{2}\right)x + 17\right] = 4\left[\left(\frac{3}{4}\right)x - \frac{5}{4}\right] \qquad \text{ref. step 2}$$

$$10x + 68 = 3x - 5$$

$$7x + 68 = -5 \qquad \text{ref. step 5}$$

$$7x = -73 \qquad \text{ref. step 6}$$

$$x = -\frac{73}{7} \text{ or } -10\frac{3}{7}$$

*Check:*

$$\frac{1}{2}\left[5\left(-\frac{73}{7}\right) + 34\right] = \frac{1}{4}\left[3\left(-\frac{73}{7}\right) - 5\right]$$

$$-\frac{73(5)}{14} + 17 = \frac{3(-73)}{28} - \frac{5}{4}$$

$$\frac{-73(5) + 17(14)}{14} = \frac{3(-73)}{28} - \frac{5}{4}$$

$$[-73(5) + 17(14)]2 = 3(-73) - 5(7)$$

$$-730 + 476 = -219 - 35$$

$$-254 = -254$$

*Example: Solve $6x + 21 < 8x + 31$.*

$$-2x + 21 < 31 \qquad \text{ref. step 5}$$

$$-2x < 10 \qquad \text{ref. step 6}$$

$$x > -5 \qquad \text{ref. step 7}$$

Note that the inequality sign has changed.

Some word problems can be modeled using linear equations or inequalities. Watch for phrases such as *greater than, less than, at least,* or *no more than,* which indicate the need for an inequality.

*Example: The YMCA wants to sell raffle tickets to raise at least $32,000. If it must pay $7,250 in expenses and prizes out of the money collected from the tickets, how many tickets priced at $25 each must the YMCA sell?*

Since the YMCA wants to raise *at least $32,000,* that means it would be happy to get *$32,000 or more.* This problem requires an inequality.

Let $x$ = number of tickets sold.

Then $25x$ = total money collected for selling $x$ tickets.

Total money minus expenses is greater than $32,000.

$$25x - 7{,}250 \geq 32{,}000$$

$$25x \geq 39{,}250$$

$$x \geq 1{,}570$$

If the YMCA sells *1,570 tickets or more,* it will raise *at least $32,000.*

*Example: The Simpsons went out for dinner. All four of them ordered the aardvark steak dinner. Bart paid for the four meals and included a tip of $12, for a total of $84.60. How much was one aardvark steak dinner?*

Let $x$ = the price of one aardvark dinner.
So $4x$ = the price of 4 aardvark dinners.
$4x = 84.60 - 12$
$4x = 72.60$
$x = \frac{72.60}{4} = \$18.15$ The price of one aardvark dinner

## SKILL 1.3 Relate the graphical and algebraic representations of linear equations or inequalities on a number line or in the coordinate plane

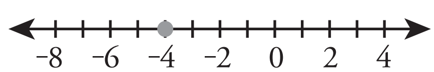
### Graphing on a Number Line

When graphing a first-degree equation on a number line, solve for the variable. The graph of the solution will be a single point on the number line. There will be no arrows.

When graphing a linear inequality, the circle on the graph will be hollow if the inequality sign is $<$ or $>$. The circle on the graph will be solid if the inequality sign is either $\geq$ or $\leq$. The arrow goes to the right for $\geq$ or $>$. The arrow goes to the left for $\leq$ or $<$.

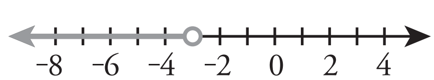
*Example: Solve 5(x + 2) + 2x = 3(x − 2).*

$5(x + 2) + 2x = 3(x - 2)$
$5x + 10 + 2x = 3x - 6$
$7x + 10 = 3x - 6$
$4x = -16$
$x = -4$




*Example: Solve 2(3x − 7) > 10x − 2.*

$2(3x - 7) > 10x - 2$
$6x - 14 > 10x - 2$
$-4x > 12$
$x < -3$ Note the change in the inequality sign when dividing by a negative number.



**SKILL 1.3 Practice 1**

Solve the following equations or inequalities. Graph the solution sets.

1. $5x - 1 > 14$
2. $7(2x - 3) + 5x = 19 - x$
3. $3x + 42 \geq 12x - 12$
4. $5 - 4(x + 3) = 9$

## Graphing in the Coordinate Plane

A **LINEAR RELATIONSHIP** between the variables $x$ and $y$ has an equation of the form **$AX + BY = C$**. To graph this equation in the coordinate plane, find either one point $(x, y)$ on the line and the slope of the line, or else find two points on the line. To find a point and the slope, solve the equation for $y$. This puts the equation in **SLOPE-INTERCEPT FORM**, **$Y = MX + B$**. The point $(0, b)$ is the $y$-intercept, and $m$ is the line's slope.

**LINEAR RELATIONSHIP:** a relationship that can be expressed by an equation of the form $ax + by = c$

**SLOPE-INTERCEPT FORM:** $y = mx + b$

To find any two points, substitute any two numbers for $x$ and then solve for each corresponding value of $y$. To find the intercepts, substitute 0 for $x$ and solve for $y$, and then substitute 0 for $y$ and solve for $x$.

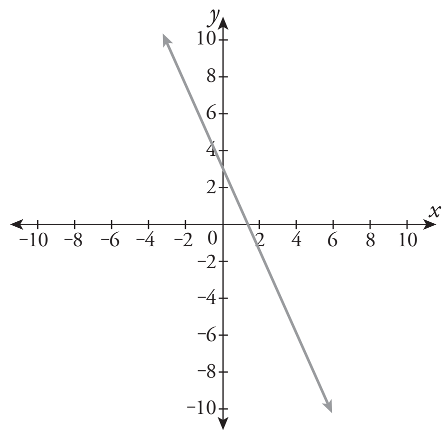
Remember that a graph will go up as it goes to the right when the slope is positive. A negative slope will make the graph go down as it goes to the right.

*A graph will go up as it goes to the right when the slope is positive. A negative slope will make the graph go down as it goes to the right.*

If the equation solves to **$x$ = any number**, then the graph is a **vertical line**. It has only an $x$-intercept, and **its slope is undefined.**

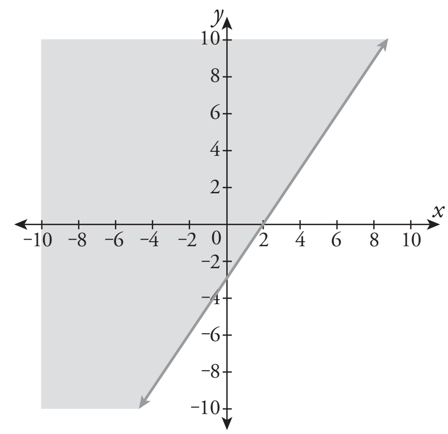
If the equation solves to **$y$ = any number**, then the graph is a **horizontal line**. It has only a $y$-intercept, and **its slope is zero.**

When graphing a linear inequality, the graph of the line will be dotted if the inequality sign is $<$ or $>$. If the inequality sign is $\geq$ or $\leq$ the graph of the line will be solid. Shade above the line when the inequality sign is $>$ or $\geq$. Shade below the line when the inequality sign is $<$ or $\leq$. For inequalities of the form $x >$, $x \leq$, $x <$, or $x \geq$ some number, draw a vertical line (solid or dotted). Shade to the right for $>$ or $\geq$, and shade to the left for $<$ or $\leq$.



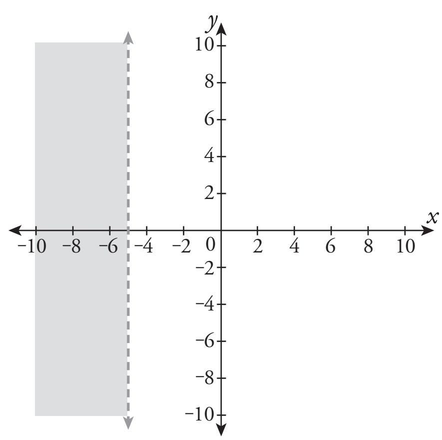
$5x + 2y = 6$

$y = -\frac{5}{2}x + 3$



$3x - 2y \geq 6$

$y \leq \frac{3}{2}x - 3$



$3x + 12 < -3$

$x < -5$

**SKILL 1.3 Practice 2**

Graph the following.

1. $2x - y = -4$
2. $x + 3y > 6$
3. $3x + 2y \leq 2y - 6$

The solution to a system of linear inequalities is represented by the part of the graph where the shading for all of the inequalities overlaps. For instance, if the graph of one inequality was shaded with red, and the graph of another inequality was shaded with blue, then the overlapping area would be shaded purple. The purple area would be the points that make up the solution set of the system.

*Example: Solve by graphing.*

$x + y \leq 6$

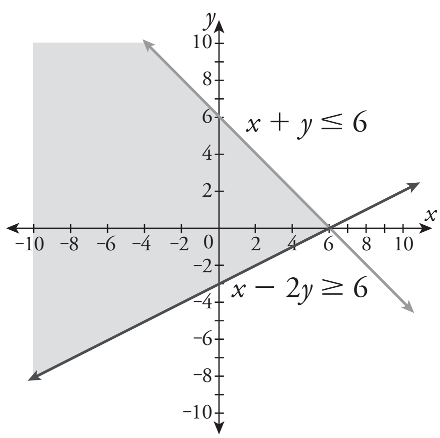
$x - 2y \leq 6$

Solving the inequalities for *y*, you see that they become:

$y \leq -x + 6$ ($y$-intercept $= 6$ and slope $= -1$)

$y \geq \frac{1}{2}x - 3$ ($y$-intercept $= -3$ and slope $= \frac{1}{2}$)

A graph with shading is shown below.




## SKILL 1.4 Determine the slope, intercepts, or equation of a line, given appropriate information

### Finding the Slope and Intercepts

To find the *y*-intercept given the equation of a line, substitute 0 for *x* and solve for *y*. The *y*-intercept is also the value of *b* when the equation is expressed in the slope-intercept form, $y = mx + b$. The slope of the line is given by *m*.

To find the *x*-intercept of a line, substitute 0 for *y* and solve for *x*.

*Example: Find the slope and intercepts of 3x + 2y = 14.*

$3x + 2y = 14$

$2y = -3x + 14$

$y = -\frac{3}{2}x + 7$

The slope of the line is $-\frac{3}{2}$, the value of *m*.

The *y*-intercept of the line is 7.

The intercepts can also be found by substituting 0 for $x$ and $y$, respectively, in the equation.

To find the $y$-intercept:

Let $x = 0$; $3(0) + 2y = 14$

$0 + 2y = 14$

$2y = 14$

$y = 7$

(0, 7) is the $y$-intercept.

To find the $x$-intercept:

Let $y = 0$; $3x + 2(0) = 14$

$3x + 0 = 14$

$3x = 14$

$x = \frac{14}{3}$

$\left(\frac{14}{3}, 0\right)$ is the $x$-intercept.

**SKILL 1.4 Practice 1**

Find the slope and the intercepts (if they exist) for the following equations.

1. $5x + 7y = -70$
2. $x - 2y = 14$
3. $5x + 3y = 3(5 + y)$
4. $2x + 5y = 15$

## Finding the Equation of a Graph or Line

The equation of a graph can be found by finding its slope and $y$-intercept. To find the slope, find two points on the graph at which coordinates are integer values. Using points $(x_1, y_1)$ and $(x_2, y_2)$:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

The $y$-intercept is the $y$-coordinate of the point where the line crosses the $y$-axis. The equation can be written in slope-intercept form, $y = mx + b$, where $m$ is the slope and $b$ is the $y$-intercept. To rewrite the equation in some other form, multiply each term by the common denominator of all the fractions (if the equation contains fractions). Then rearrange terms as necessary.

*Remember that $(x_a, y_a)$ can be $(x_1, y_1)$ or $(x_2, y_2)$. If m, the value of the slope, is distributed through the parentheses, the equation can be rewritten in other forms.*

To find the equation of a line given two points on its graph, first find the slope of the line using the method described above. The slope is denoted by the letter $m$. Then, to write the equation of the line, choose either of the given points. Substitute its coordinates into the point-slope form of a linear equation:

$$Y - y_a = m(X - x_a)$$

Remember that $(x_a, y_a)$ can be $(x_1, y_1)$ or $(x_2, y_2)$. If $m$, the value of the slope, is distributed through the parentheses, the equation can be rewritten in other forms.

*Example: Find the equation of the line through (9, −6) and (−1, 2).*

The slope $m = \frac{2-(-6)}{-1-9} = \frac{8}{-10} = -\frac{4}{5}$.

$Y - y_a = m(X - x_a) \rightarrow Y - 2 = \left(-\frac{4}{5}\right)(X - (-1)) \rightarrow$

$Y - 2 = \left(-\frac{4}{5}\right)(X + 1) \rightarrow Y - 2 = \left(-\frac{4}{5}\right)X - \frac{4}{5} \rightarrow$

$Y = -\frac{4}{5}x + \frac{6}{5}$ This is the slope-intercept form.

Multiplying by 5 to eliminate fractions, we can find the standard form.

$5Y = -4X + 6 \rightarrow 4X + 5Y = 6$ Standard form

**SKILL 1.4 Practice 2**

Write the equation of the line through the two points.

1. (5, 8) and (−3, 2)
2. (11, 10) and (11, −3)
3. (−4, 6) and (6, 12)
4. (7, 5) and (−3, 5)

## SKILL 1.5 Formulate and solve systems of linear equations or inequalities, including models of real-world situations

Word problems can sometimes be solved by creating a system of two or more equations with as many unknowns. This system can then be solved by using **substitution**, the **addition-subtraction method**, or **determinants**.

*Example: Farmer Greenjeans bought 4 cows and 6 sheep for $1,700. Mr. Ziffel bought 3 cows and 12 sheep for $2,400. If all the cows were the same price and all the sheep were another price, find the price charged for a cow and for a sheep.*

Let $x$ = price of a cow.

Let $y$ = price of a sheep.

Then Farmer Greenjeans' equation would be: $4x + 6y = 1{,}700$

Mr. Ziffel's equation would be: $3x + 12y = 2{,}400$

## Solving by Addition-Subtraction

We can solve the Farmer Greenjeans problem by the addition-subtraction method as follows:

Multiply the first equation by −2: $-2(4x + 6y = 1{,}700)$

Keep the other equation the same: $3x + 12y = 2{,}400$

The equations can be added to each other to eliminate one variable and solve for the other variable.

$$\begin{array}{rl} -8x - 12y &= -3{,}400 \\ 3x + 12y &= 2{,}400 \\ \hline -5x \phantom{+ 12y} &= -1{,}000 \end{array} \quad \text{Add these equations.}$$

$x = 200 \leftarrow$ The price of a cow is \$200.

Solving for $y$, $y = 150$. ← The price of a sheep is \$150.

## Solving by Substitution

We can solve the Farmer Greenjeans problem by the substitution method as follows:

Solve one of the equations for one of the variables. (Eliminate fractions if possible.) Substitute this expression into the equation that you have not yet used. Then solve the resulting equation for the value of the remaining variable.

$4x + 6y = 1{,}700$

$3x + 12y = 2{,}400 \leftarrow$ Solve this equation for $x$.

The second equation becomes $x = 800 - 4y$. Now substitute $800 - 4y$ in place of $x$ in the first equation, $4x + 6y = 1{,}700$.

$$\begin{aligned} 4(800 - 4y) + 6y &= 1{,}700 \\ 3{,}200 - 16y + 6y &= 1{,}700 \\ 3{,}200 - 10y &= 1{,}700 \\ -10y &= -1{,}500 \\ y &= 150, \text{ or \$150 for a sheep.} \end{aligned}$$

Substitute 150 for $y$ in either equation to find $x$.

$$\begin{aligned} 4x + 6(150) &= 1{,}700 \\ 4x + 900 &= 1{,}700 \\ 4x &= 800, \text{ so } x = 200, \text{ or \$200 for a cow.} \end{aligned}$$

## Solving by Determinants

We can solve the Farmer Greenjeans problem using determinants as follows:

Let $x$ = price of a cow.

Let $y$ = price of a sheep.

Then Farmer Greenjeans' equation would be: $4x + 6y = 1{,}700$
Mr. Ziffel's equation would be: $3x + 12y = 2{,}400$

Divide one 2-by-2 determinant by another 2-by-2 determinant. The bottom determinant is filled with the *x* and *y* terms' coefficients. The top determinant is almost the same as the bottom determinant. The only difference is that when you are solving for *x,* the *x* coefficients are replaced with the constants. Likewise, when you are solving for *y,* the *y* coefficients are replaced with the constants. The value of the 2-by-2 determinant $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is found by calculating $ad - bc$.

$$x = \frac{\begin{pmatrix} 1{,}700 & 6 \\ 2{,}400 & 12 \end{pmatrix}}{\begin{pmatrix} 4 & 6 \\ 3 & 12 \end{pmatrix}} = \frac{1{,}700(12) - 6(2{,}400)}{4(12) - 6(3)} = \frac{20{,}400 - 14{,}400}{48 - 18} = \frac{6{,}000}{30} = 200$$

$$y = \frac{\begin{pmatrix} 4 & 1{,}700 \\ 3 & 2{,}400 \end{pmatrix}}{\begin{pmatrix} 4 & 6 \\ 3 & 12 \end{pmatrix}} = \frac{2{,}400(4) - 3(1{,}700)}{4(12) - 6(3)} = \frac{9{,}600 - 5{,}100}{48 - 18} = \frac{4{,}500}{30} = 150$$

NOTE: The bottom determinant is always the same, whether you are solving for *x* or for *y.*

## More Real-World Examples

*Example: Mrs. Allison bought 1 pound of potato chips, a 2-pound roast, and 3 pounds of apples for a total of \$8.19. Mr. Bromberg bought a 3-pound roast and 2 pounds of apples for \$9.05. Karina Kaufman bought 2 pounds of potato chips, a 3-pound roast, and 5 pounds of apples for \$13.25. Find the per-pound price of each item.*

Let $x$ = price of a pound of potato chips.
Let $y$ = price of a pound of roast.
Let $z$ = price of a pound of apples.
Mrs. Allison's equation would be: $1x + 2y + 3z = 8.19$
Mr. Bromberg's equation would be: $3y + 2z = 9.05$
K. Kaufman's equation would be: $2x + 3y + 5z = 13.25$

To solve by **substitution**:

Take the first equation and solve it for *x.* (This equation was chosen because *x* is the easiest variable to isolate in this set of equations.) The equation becomes:

$x = 8.19 - 2y - 3z$

Substitute this expression into the other equations in place of the letter *x*:

$3y + 2z = 9.05 \leftarrow$ equation 2
$2(8.19 - 2y - 3z) + 3y + 5z = 13.25 \leftarrow$ equation 3

Simplify the equations by combining like terms:

$3y + 2z = 9.05 \leftarrow$ equation 2
$-1y - 1z = -3.13 \leftarrow$ equation 3