DOMAIN I. MATHEMATICS

COMPETENCY 0001 UNDERSTAND PRINCIPLES , CONCEPTS, AND PROCEDURES RELATED TO REPRESENTATION AND COMMUNICATION.

Skill 1.1 Extracting mathematical information from a variety of sources (e.g., pictures, diagrams, text, graphs)

Often data is made more readable and user-friendly by consolidating the information in the form of a graph.

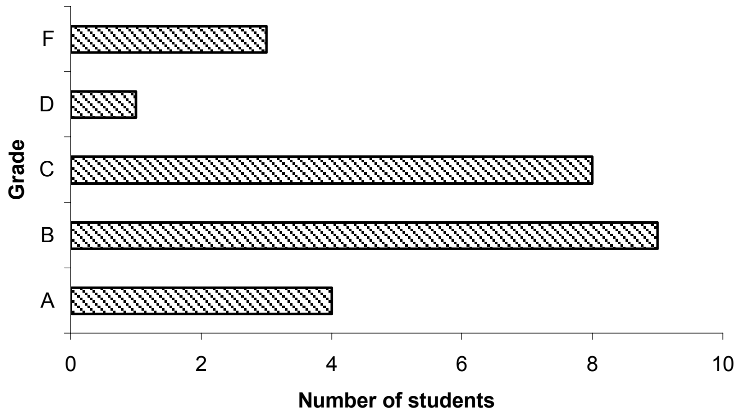
Bar graphs are used to compare various quantities using bars of different lengths. A **pictograph** shows comparison of quantities using symbols. Each symbol represents a number of items. To make a **bar graph** or a **pictograph**, determine the scale to be used for the graph. Then determine the length of each bar on the graph or determine the number of pictures needed to represent each item of information. Be sure to include an explanation of the scale in the legend.

Example: A class had the following grades:
4 A's, 9 B's, 8 C's, 1 D, 3 F's.
Graph these on a bar graph and a pictograph.

Pictograph

Grade	Number of Students
A	☺☺☺☺
B	☺☺☺☺☺☺☺☺☺
C	☺☺☺☺☺☺☺☺
D	☺
F	☺☺☺

Bar graph



Line graphs are used to show trends, often over a period of time.

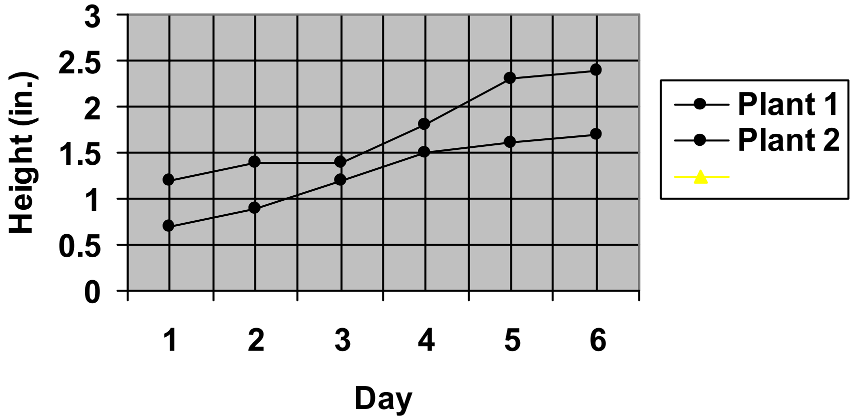
To make a line graph, determine appropriate scales for both the vertical and horizontal axes (based on the information to be graphed). Describe what each axis represents and mark the scale periodically on each axis. Graph the individual points of the graph and connect the points on the graph from left to right.

<u>Example:</u> Graph the following information using a line graph.

Height of Two Pea Plants for Six Days

Day	**1**	**2**	**3**	**4**	**5**	**6**
Plant 1 Height (in.)	1.2	1.4	1.4	1.8	2.3	2.4
Plant 2 Height (in.)	0.7	0.9	1.2	1.5	1.6	1.7

Height of Two Pea Plants for Six Days



Circle graphs show the relationship of various parts of a data set to each other and the whole. Each part is shown as a percentage of the total and occupies a proportional sector of the circular area. To make a circle graph, total all the information that is to be included on the graph. Determine the central angle to be used for each sector of the graph using the following formula:

$$\frac{\text{information}}{\text{total information}} \times 360^\circ = \text{degrees in central} \square$$

Lay out the central angles according to these sizes, label each section and include its percentage.

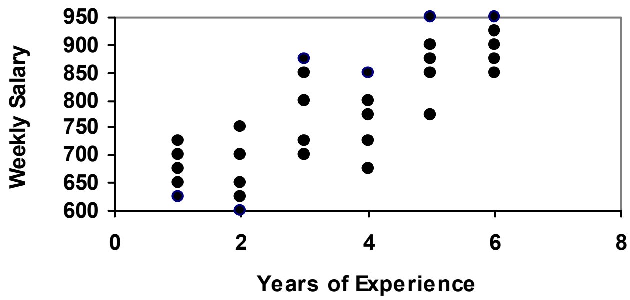
<u>Example:</u> Graph this information on a circle graph:

Monthly expenses:

Rent, $400
Food, $150
Utilities, $75
Clothes, $75
Church, $100
Misc., $200

[redacted]

Scatter plots compare two characteristics of the same group of things or people and usually consist of a large body of data. They show how much one variable is affected by another. The relationship between the two variables is their **correlation**. The closer the data points come to making a straight line when plotted, the closer the correlation.



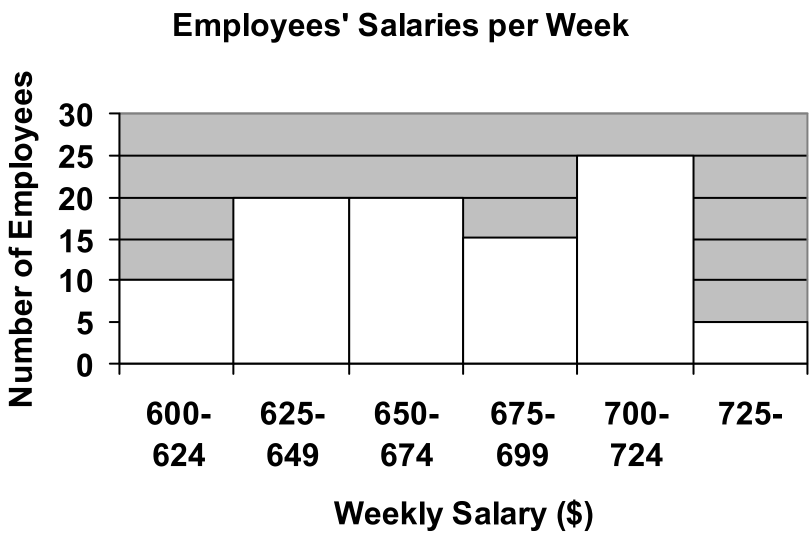
Stem-and-leaf plots are visually similar to histograms. The **stems** are the digits in the greatest place value of the data values, and the **leaves** are the digits in the next greatest place values. Stem and leaf plots are best suited for small sets of data and are especially useful for comparing two sets of data.

Example: Make a stem-and-leaf plot to display the following test scores: 49, 54, 59, 61, 62, 63, 64, 66, 67, 68, 68, 70, 73, 74, 76, 76, 76, 77, 77, 77, 77, 78, 78, 78, 78, 83, 85, 85, 87, 88, 90, 90, 93, 94, 95, 100, 100.

stem	leaves
4	9
5	4 9
6	1 2 3 4 6 7 8 8
7	0 3 4 6 6 6 7 7 7 7 8 8 8 8
8	3 5 5 7 8
9	0 0 3 4 5
10	0 0

Histograms are used to summarize information from large sets of data that can be naturally grouped into intervals. The vertical axis indicates **frequency** (the number of times any particular data value occurs), and the horizontal axis indicates data values or ranges of data values. The number of data values in any interval is the **frequency of the interval**.

Example: The human resources department of a small company surveyed workers on their weekly salaries. Ten workers earned between $600 and $624 per week. Twenty workers earned between $625–$649/week. Twenty workers earned between $650–$674/week. Fifteen workers earned between $675–$699/week. Twenty-five workers earned between $700–$724/week. Five workers earned $725 or more each week. Draw a histogram to display the results.



Example: Justin surveyed his classmates on their favorite magazines. What is the best graph to show his data?

Since Justin is using counting data, so a bar graph or pictograph would be the best format to display his data.

Example: Lakeisha recorded the amount of rain that fell in Augusta, GA, for 6 weeks. What is the best graph to show her data?

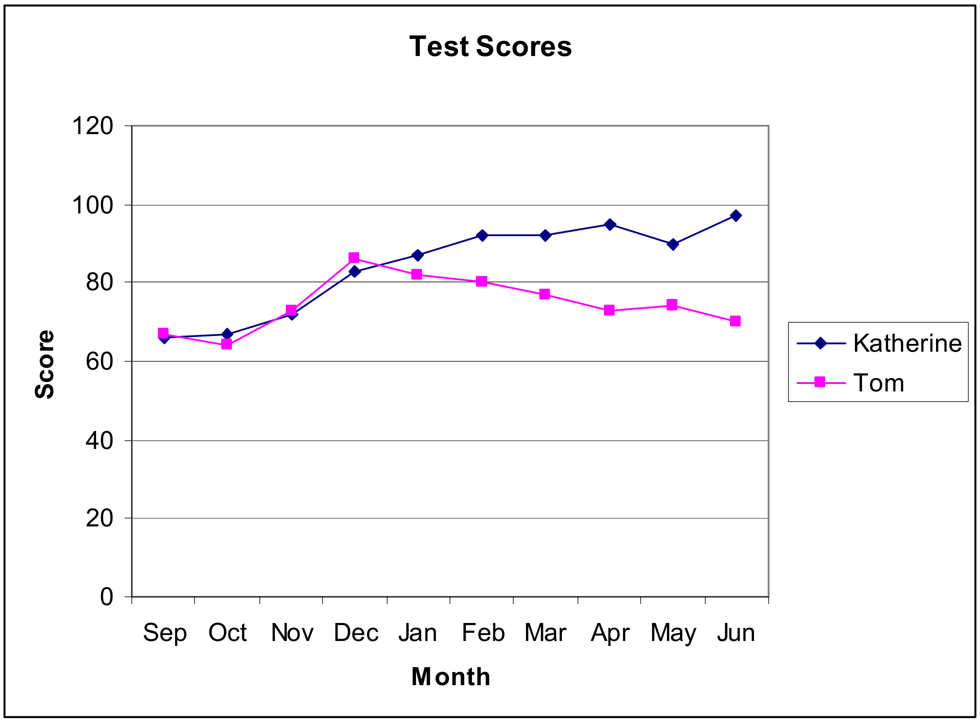
Since Lakeisha is displaying a change over time, a line graph would be the best format to show her data.

Skill 1.2 Using textual, graphic, numeric, and symbolic representations to communicate mathematical concepts

Displaying data in graphical format can reveal a lot of information about the data set. An **inference** is a statement that is derived from reasoning. When reading a graph, inferences help with interpretation of the data that is being presented. From this information, a **conclusion** and even **predictions** about what the data actually means is possible.

A **trend** line on a line graph shows the correlation between two sets of data. A trend may show positive correlation (both sets of data get greater together) negative correlation (one set of data increases while the other decreases), or no correlation.

Example: Katherine and Tom were both doing poorly in math class. Their teacher had a conference with each of them in November. The following graph shows their math test scores during the school year.



What kind of trend does this graph show?

This graph shows that there is a positive trend in Katherine's test scores and a negative trend in Tom's test scores.

What inferences can you make from this graph?

We can infer that Katherine's test scores rose steadily after November. Tom's test scores spiked in December but then began a negative trend.

What conclusion can you draw based upon this graph?

We can conclude that Katherine took her teacher's meeting seriously and began to study in order to do better on the exams. It seems as though Tom tried harder for a bit but his test scores eventually slipped back down to the level where he began.

Skill 1.3 Translating between textual, graphic, numeric, and symbolic representations

Mathematics is, in some ways, a formalization of language that concerns such concepts as quantity and organization. Mathematics often involves symbolic representations, which can help alleviate the ambiguities found in common language. Naturally, then, communication of mathematical ideas requires conversion back and forth from verbal and symbolic forms. These two forms can often help to elucidate one another when an attempt is made to understand an idea that they represent. Other representations of mathematical concepts can involve graphs and numerals.

Mathematical ideas and expressions can sometimes be easily translated into language; for instance, basic arithmetic operations are usually fairly easy to express in everyday language (although complicated expressions may be less so). In some cases, common language more easily expresses certain ideas than does symbolic language (and sometimes vice versa). Much of the translation process is learned through practicing expression of mathematical ideas in verbal (or written) form and by translating verbal or written expressions into a symbolic form.

Likewise, the use of graphic or numeric representations may offer a more lucid presentation. For instance, although a function can be represented symbolically using a polynomial of some degree, the polynomial may be so complex that a graph of the function provides a better and simpler description. Likewise, numbers—especially those with decimals—are more easily expressed using numerals (1, 2, 3…) than using words (one, two, three…). As such, translation among these various representations can help improve and simplify mathematical communication.

The material throughout this guide attempts to present mathematical ideas both in a variety of formats. Thus, practicing by carefully following the text and example problems and by attempting to articulate the various concepts using English, using mathematical symbols (including numerals), and using graphs should help the student (and teacher) of mathematics gain mastery of this skill.

Example: Express the following in written language (textual) form: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

This symbolic expression, in written language form, is simply “the set of integers.”

Throughout this guide, mathematical operations and situations are represented through words, algebraic symbols, geometric diagrams and graphs. A few commonly used representations are discussed below.

The basic mathematical operations include addition, subtraction, multiplication and division. In word problems, these are represented by the following typical expressions.

Operation	Descriptive Words
Addition	"plus", "combine", "sum", "total", "put together"
Subtraction	"minus", "less", "take away", "difference"
Multiplication	"product", "times", "groups of"
Division	"quotient", "into", "split into equal groups",

Some verbal and symbolic representations of basic mathematical operations include the following:

7 added to a number	n + 7
a number decreased by 8	n – 8
12 times a number divided by 7	12n ÷ 7
28 less than a number	n – 28
the ratio of a number to 55	$\frac{n}{55}$
4 times the sum of a number and 21	4(n + 21)

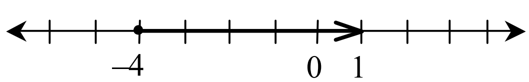
Multiplication can be shown using arrays. For instance, 3 × 4 can be expressed as 3 rows of 4 each

In a similar manner, addition and subtraction can be demonstrated with symbols.

ψ ψ ψ ξ ξ ξ ξ
3 + 4 = 7
7 – 3 = 4

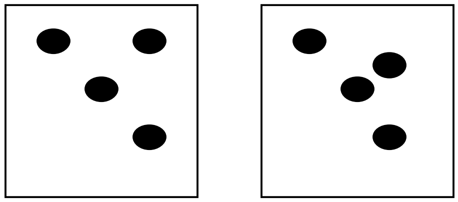
Fractions can be represented using pattern blocks, fraction bars, or paper folding.

Diagrams of arithmetic operations can present mathematical data in visual form. For example, a number line can be used to add and subtract, as illustrated below.

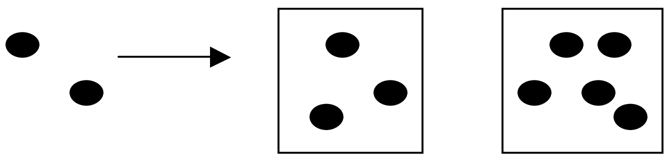


Five added to negative four on the number line or –4 + 5 = 1.

Pictorial representations can also be used to explain the arithmetic processes.



The diagram above shows two groups of four equal eight, or 2 x 4 = 8. The next diagram illustrates addition of two objects to three objects, resulting in five objects.



SEE also Skills 1.1 and 1.2

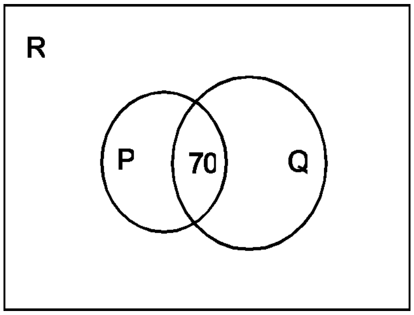
Skill 1.4 Using mathematical language to communicate ideas and information, including interpreting mathematical terminology, symbols, and representations

Examples, illustrations, and symbolic representations are useful tools in explaining and understanding mathematical concepts. The ability to create examples and alternative methods of expression allows students to solve real world problems and better communicate their thoughts. Many different kinds of graphs, diagrams, symbols and tables have been used throughout this guide.

Concrete examples are real world applications of mathematical concepts. For example, measuring the shadow produced by a tree or building is a real world application of trigonometric functions; acceleration or velocity of a car is an application of derivatives; and finding the volume or area of a swimming pool is a real world application of geometric principles.

Pictorial illustrations of mathematic concepts help clarify difficult ideas and simplify problem solving.

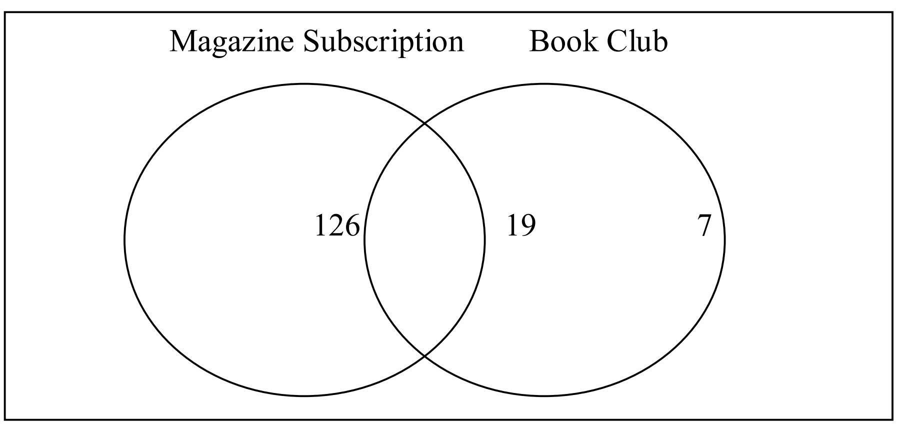
Example: Rectangle R represents the 300 students in School A. Circle P represents the 150 students that participated in band. Circle Q represents the 170 students that participated in a sport. 70 students participated in both band and a sport.



Pictorial representation of above situation.

Example: A marketing company surveyed 200 people and found that 145 people subscribed to at least one magazine, 26 people subscribed to at least one book-of-the-month club, and 19 people subscribed to at least one magazine and one book club. How many people did not subscribe to either a magazine or book club?

Draw a Venn diagram.



200 – (126 + 19 + 7) = 48 people do not subscribe to either.

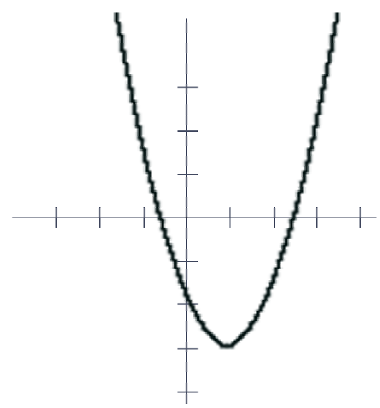
Symbolic representation is the basic language of mathematics. Converting data to symbols allows for easy manipulation and problem solving. Students should have the ability to recognize what the symbolic notation represents and convert information into symbolic form. For example, from the graph of a line, students should have the ability to determine the slope and intercepts and derive the line's equation from the observed data. Another possible application of symbolic representation is the formulation of algebraic expressions and relations from data presented in word problem form.

Mathematical concepts and procedures can take many different forms. Students of mathematics must be able to recognize different forms of equivalent concepts.

For example, we can represent the slope of a line graphically, algebraically, verbally, and numerically. A line drawn on a coordinate plane will show the slope. In the equation of a line, y = mx + b, the term m represents the slope. We can define the slope of a line several different ways. The slope of a line is the change in the value of the y divided by the change in the value of x over a given interval. Alternatively, the slope of a line is the ratio of "rise" to "run" between two points. Finally, we can calculate the numeric value of the slope by using the verbal definitions and the algebraic representation of the line.

Most mathematical concepts can be expressed in multiple ways. For example a parabola can be expressed as an equation or a graph. A function can be rewritten as a table. Each way is an equally accurate method of representing the concept, but different techniques may be useful in different situations. It is therefore important to be able to translate any concept into the most appropriate form for addressing any given problem.

$y = (x-1)^2 - 3$ *is equivalent to*



One example of a mathematical idea that can be presented in multiple ways is any group of numbers that is a subset of the real number line. There are three common ways to denote such a set of numbers: graphically on the real number line, in interval notation, and in set notation. For example, the set of numbers consisting of all values greater than negative 2 and less than or equal to 4 could be written in the following three ways:

Graphically:
In this type of notation, open circles exclude an endpoint while closed circles include it.

Interval Notation: (-2, 4]. Interval notation uses round brackets to exclude the endpoint of a set, and square brackets to include it.

Set Notation: $\{-2 < x \leq 4\}$

Depending on why you need to represent this group of numbers, each form of notation has its advantages. Set notation is the most flexible, since finite sets can be written as lists within the brackets. However, it is often far more cumbersome than interval notation, and in some circumstances graphical notation may be the most clear.

An algebraic formula is an equation that describes a relationship among variables. While it is not often necessary to derive the formula, one must know how to rewrite a given formula in terms of a desired variable.

<u>Example:</u> Given that the relationship of voltage, V, applied across a material with electrical resistance, R, when a current, I, is flowing through the material is given by the formula $V = IR$. Find the resistance of the material when a current of 10 milliamps is flowing, when the applied voltage is 2 volts.

$V = IR$. Solve for R.
$IR = V$; $R = V/I$ Divide both sides by I.
When $V = 2$ volts; $I = 10\times10^{-3}$ amps;

$$R = \frac{2}{10^1\times10^{-3}}$$

$$R = \frac{2}{10^{-2}}$$ Substituting in $R = V/I$, we get,

$$R = 2\times10^2$$

$$R = 200 \text{ ohms}$$

Another example of translating between mathematical language and everyday language is the conversion of recipes to different serving sizes. The conversion factor, the number we multiply each ingredient by, is:

$$\text{Conversion Factor} = \frac{\text{Number of Servings Needed}}{\text{Number of Servings in Recipe}}$$

Example: Consider the following recipe.

3 cups flour
½ tsp. baking powder
2/3 cups butter
2 cups sugar
2 eggs

If the above recipe serves 8, how much of each ingredient do we need to serve only 4 people?
First, determine the conversion factor.

$$\text{Conversion Factor} = \frac{4}{8} = \frac{1}{2}$$

Next, multiply each ingredient by the conversion factor.

3 x ½ =	1 ½ cups flour
½ x ½ =	¼ tsp. baking powder
2/3 x ½ = 2/6 =	1/3 cups butter
2 x ½ =	1 cup sugar
2 x ½ =	1 egg

COMPETENCY 0002 UNDERSTAND PRINCIPLES, CONCEPTS, AND PROCEDURES RELATED TO MATHEMATICAL REASONING, PROOF, AND CONNECTIONS.

Skill 2.1 Applying knowledge of formal and informal mathematical reasoning processes (e.g., using logical reasoning to draw and justify conclusions)

Deductive reasoning

Deductive thinking is the process of arriving at a conclusion based on other statements that are all known to be true, such as theorems, axioms, or postulates. Conclusions found by deductive thinking based on true statements will *always* be true.

Inductive reasoning

Inductive thinking is the process of finding a pattern from a group of examples. The pattern is the conclusion that a set of examples seemed to indicate. It may be a correct conclusion or it may be an incorrect conclusion, as other examples may not follow the predicted pattern.

<u>Example:</u>

Suppose:
On Monday Mr.Peterson eats breakfast at McDonalds.
On Tuesday Mr.Peterson eats breakfast at McDonalds.
On Wednesday Mr.Peterson eats breakfast at McDonalds.
On Thursday Mr.Peterson eats breakfast at McDonalds again.

Conclusion: On Friday Mr. Peterson will eat breakfast at McDonalds again.

This is a conclusion based on inductive reasoning. Based on several days' observations, you conclude that Mr. Peterson will eat at McDonalds. This may or may not be true, but it is a valid inductive conclusion.

Adaptive reasoning

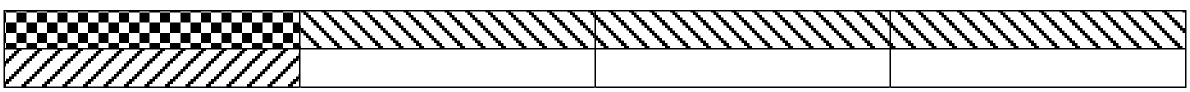
A **valid argument** is a statement made about a pattern or relationship between elements, thought to be true, which is subsequently justified through repeated examples and logical reasoning. Another term for a valid argument is a **proof**.

For example, the statement that the sum of two odd numbers is always even could be tested through actual examples.

Two Odd Numbers	Sum	Validity of Statement
1+1	2 (even)	Valid
1+3	4 (even)	Valid
61+29	90 (even)	Valid
135+47	182 (even)	Valid
253+17	270 (even)	Valid
1,945+2,007	3,952 (even)	Valid
6,321+7,851	14,172 (even)	Valid

Adding two odd numbers always results in a sum that is even. It is a valid argument based on the justifications in the table above.

Consider another example. The statement that a fraction of a fraction can be determined by multiplying the numerator by the numerator and the denominator by the denominator can be proven through logical reasoning. For example, one-half of one-quarter of a candy bar can be found by multiplying ½ x ¼. The answer would be one-eighth. The validity of this argument can be demonstrated as valid with a model.



The entire rectangle represents one whole candy bar. The top half section of the model is shaded in one direction to demonstrate how much of the candy bar remains from the whole candy bar. The left quarter, shaded in a different direction, demonstrates that one-quarter of the candy bar has been given to a friend. Since the whole candy bar is not available to give out, the area that is double-shaded is the fractional part of the ½ candy bar that has been actually given away. That fractional part is one-eighth of the whole candy bar, as shown in both the sketch and the algorithm.

Skill 2.2 Developing and evaluating conjectures and informal proofs

Conditional statements are frequently written in "**if-then**" form. The "if" clause of the conditional is known as the **hypothesis**, and the "then" clause is called the **conclusion**. In a proof, the hypothesis is the information that is assumed to be true, while the conclusion is what is to be proven true. A conditional is considered to be of the form:

If p, then q
p is the hypothesis and q is the conclusion.