## COMPETENCY 0001 UNDERSTAND MATHEMATICAL PROBLEM SOLVING AND THE CONNECTIONS BETWEEN AND AMONG THE FIELDS OF MATHEMATICS AND OTHER DISCIPLINES.

Successful math teachers introduce their students to multiple problem solving strategies and create a classroom environment where free thought and experimentation are encouraged. Teachers can promote problem solving by allowing multiple attempts at problems, giving credit for reworking test or homework problems, and encouraging the sharing of ideas through class discussion. There are several specific problem solving skills with which teachers should be familiar.

The **guess-and-check** strategy calls for students to make an initial guess at the solution, check the answer, and use the outcome of to guide the next guess. With each successive guess, the student should get closer to the correct answer. Constructing a table from the guesses can help organize the data.

Example:

There are 100 coins in a jar. 10 are dimes. The rest are pennies and nickels. There are twice as many pennies as nickels. How many pennies and nickels are in the jar?

There are 90 total nickels and pennies in the jar (100 coins – 10 dimes).

There are twice as many pennies as nickels. Make guesses that fulfill the criteria and adjust based on the answer found. Continue until we find the correct answer, 60 pennies and 30 nickels.

| Number of Pennies | Number of Nickels | Total Number of Pennies and Nickels |
|---|---|---|
| 40 | 20 | 60 |
| 80 | 40 | 120 |
| 70 | 35 | 105 |
| 60 | 30 | 90 |

When solving a problem where the final result and the steps to reach the result are given, students must **work backwards** to determine what the starting point must have been.

Example:

John subtracted seven from his age, and divided the result by 3. The final result was 4. What is John's age?

> Work backward by reversing the operations.
> 4 x 3 = 12;
> 12 + 7 = 19
> John is 19 years old.

**Estimation** and testing for **reasonableness** are related skills students should employ both before and after solving a problem. These skills are particularly important when students use calculators to find answers.

Example:

Find the sum of 4387 + 7226 + 5893.

| | |
|---|---|
| 4300 + 7200 + 5800 = 17300 | Estimation. |
| 4387 + 7226 + 5893 = 17506 | Actual sum. |

By comparing the estimate to the actual sum, students can determine that their answer is reasonable.

The **questioning technique** is a mathematic process skill in which students devise questions to clarify the problem, eliminate possible solutions, and simplify the problem solving process. By developing and attempting to answer simple questions, students can tackle difficult and complex problems.

The use of supplementary materials in the classroom can greatly enhance the learning experience by stimulating student interest and satisfying different learning styles. Manipulatives, models, and technology are examples of tools available to teachers.

**Manipulatives** are materials that students can physically handle and move. Manipulatives allow students to understand mathematic concepts by allowing them to see concrete examples of abstract processes. Manipulatives are attractive to students because they appeal to the students' visual and tactile senses. Available for all levels of math, manipulatives are useful tools for reinforcing operations and concepts. They are not, however, a substitute for the development of sound computational skills.

**Models** are another means of representing mathematical concepts by relating the concepts to real-world situations.
Teachers must choose wisely when devising and selecting models because, to be effective, models must be applied properly. For example, a building with floors above and below ground is a good model for introducing the concept of negative numbers. It would be difficult, however, to use the building model in teaching subtraction of negative numbers.

Finally, there are many forms of **technology** available to math teachers. For example, students can test their understanding of math concepts by working on skill specific computer programs and websites. Graphing calculators can help students visualize the graphs of functions. Teachers can also enhance their lectures and classroom presentations by creating multimedia presentations.

**Example – Life Science**

Examine an animal population and vegetation density in a biome over time.

**Example – Physical Science**

Explore motions and forces by calculating speeds based on distance and time traveled and creating a graph to represent the data.

**Example – Geography**

Explore and illustrate knowledge of earth landforms.

**Example – Economics/Finance**

Compare car buying with car leasing by graphing comparisons and setting up monthly payment schedules based on available interest rates.

Mathematics dates back before recorded history. Prehistoric cave paintings with geometrical figures and slash counting have been dated prior to 20,000 BC in Africa and France. The major early uses of mathematics were for astronomy, architecture, trading and taxation.

The early history of mathematics is found in Mesopotamia (Sumeria and Babylon), Egypt, Greece and Rome. Noted mathematicians from these times include Euclid, Pythagoras, Apollonius, Ptolemy and Archimedes.

Islamic culture from the 6th through 12th centuries drew from areas ranging from Africa and Spain to India. Through India, they also drew on China. This mix of cultures and ideas brought about developments in many areas, including the concept of algebra, our current numbering system, and major developments in algebra with concepts such as zero. India was the source of many of these developments. Notable scholars of this era include Omar Khayyam and Muhammad al-Khwarizmi.

Counting boards have been found in archeological digs in Babylonia and Greece. These include the Chinese abacus whose current form dates from approximately 1200 AD. Prior to the development of the zero, a counting board or abacus was the common method used for all types of calculations.

Abelard and Fibonacci brought Islamic texts to Europe in the 12th century. By the 17th century, major new works appeared from Galileo and Copernicus (astronomy), Newton and Leibniz (calculus), and Napier and Briggs (logarithms). Other significant mathematicians of this era include René Descartes, Carl Gauss, Pierre de Fermat, Leonhard Euler and Blaise Pascal.

The growth of mathematics since 1800 has been enormous, and has affected nearly every area of life. Some names significant in the history of mathematics since 1800 (and the work they are most known for):

Joseph-Louis Lagrange (theory of functions and of mechanics)
Pierre-Simon Laplace (celestial mechanics, probability theory)
Joseph Fourier (number theory)
Lobachevsky and Bolyai (non-Euclidean geometry)
Charles Babbage (calculating machines, origin of the computer)
Lady Ada Lovelace (first known program)
Florence Nightingale (nursing, statistics of populations)
Bertrand Russell (logic)
James Maxwell (differential calculus and analysis)
John von Neumann (economics, quantum mechanics and game theory)
Alan Turing (theoretical foundations of computer science)
Albert Einstein (theory of relativity)
Gustav Roch (topology)

**COMPETENCY 0002 UNDERSTAND THE PRINCIPLES AND PROCESSES OF MATHEMATICAL REASONING.**

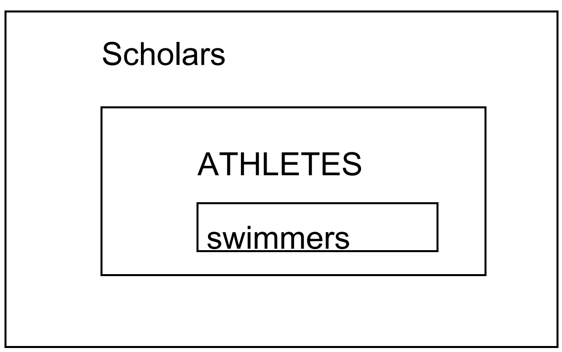
**Evaluate Arguments**

Conditional statements can be diagrammed using a **Venn diagram**. A diagram can be drawn with one circle inside another circle. The inner circle represents the hypothesis. The outer circle represents the conclusion. If the hypothesis is taken to be true, then you are located inside the inner circle. If you are located in the inner circle then you are also inside the outer circle, so that proves the conclusion is true. Sometimes that conclusion can then be used as the hypothesis for another conditional, which can result in a second conclusion.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All athletes are scholars.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are an athlete, then you are a scholar.

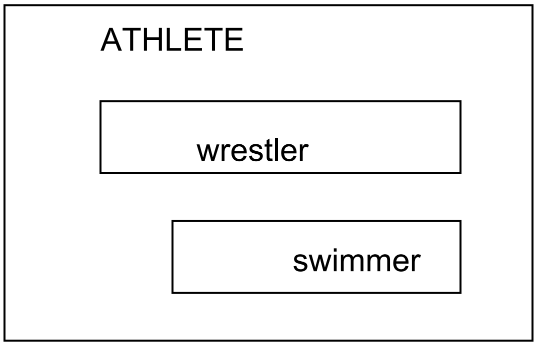



Clearly, if you are a swimmer, then you are also an athlete. This includes you in the group of scholars.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All swimmers are athletes.
All wrestlers are athletes.

In "if-then" form, these would be:
If you are a swimmer, then you are an athlete.
If you are a wrestler, then you are an athlete.




Clearly, if you are a swimmer or a wrestler, then you are also an athlete. This does NOT allow you to come to any other conclusions.

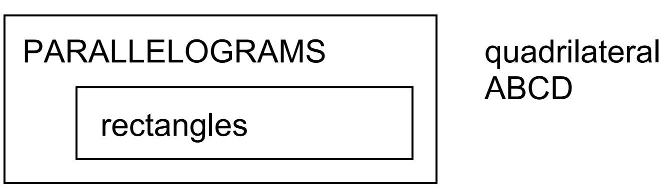
A swimmer may or may NOT also be a wrestler. Therefore,
NO CONCLUSION IS POSSIBLE.

Suppose that these statements were given to you, and you are asked to try to reach a conclusion. The statements are:

All rectangles are parallelograms.
Quadrilateral ABCD is not a parallelogram.

In "if-then" form, the first statement would be:
If a figure is a rectangle, then it is also a parallelogram.

Note that the second statement is the negation of the conclusion of statement one. Remember also that the contrapositive is logically equivalent to a given conditional. That is, **"If □ q, then □ p**". Since "ABCD is NOT a parallelogram " is like saying **"If □ q,"** then you can come to the conclusion **"then □ p".** Therefore, the conclusion is ABCD is not a rectangle. Looking at the Venn diagram below, if all rectangles are parallelograms, then rectangles are included as part of the parallelograms. Since quadrilateral ABCD is not a parallelogram, it is excluded from anywhere inside the parallelogram box. This allows you to conclude that ABCD cannot be a rectangle either.



Try These. (The answers are in the Answer Key to Practice Problems):

What conclusion, if any, can be reached? Assume each statement is true, regardless of any personal beliefs.

1. If the Red Sox win the World Series, I will die.
   I died.

2. If an angle's measure is between 0° and 90°, then the angle is acute. Angle B is not acute.

3. Students who do well in geometry will succeed in college.
   Annie is doing extremely well in geometry.

4. Left-handed people are witty and charming.
   You are left-handed.

**Algebraic Postulates for Proofs**

The following algebraic postulates are frequently used as reasons for statements in 2 column geometric proofs:

Addition Property:

If $a = b$ and $c = d$, then $a + c = b + d$.

Subtraction Property:

If $a = b$ and $c = d$, then $a - c = b - d$.

Multiplication Property:

If $a = b$ and $c \neq 0$, then $ac = bc$.

Division Property:

If $a = b$ and $c \neq 0$, then $a/c = b/c$.

| | |
|---|---|
| Reflexive Property: | $a = a$ |
| Symmetric Property: | If $a = b$, then $b = a$. |
| Transitive Property: | If $a = b$ and $b = c$, then $a = c$. |
| Distributive Property: | $a(b + c) = ab + ac$ |
| Substitution Property: | If $a = b$, then $b$ may be substituted for $a$ in any other expression ($a$ may also be substituted for $b$). |

**Write Direct and Indirect Proofs.**

In a 2 column proof, the left side of the proof should be the given information, or statements that could be proved by deductive reasoning. The right column of the proof consists of the reasons used to determine that each statement to the left was verifiably true. The right side can identify given information, or state theorems, postulates, definitions or algebraic properties used to prove that particular line of the proof is true.

Assume the opposite of the conclusion. Keep your hypothesis and given information the same. Proceed to develop the steps of the proof, looking for a statement that contradicts your original assumption or some other known fact. This contradiction indicates that the assumption you made at the beginning of the proof was incorrect; therefore, the original conclusion has to be true.

**Estimation and Approximation to Check Reasonableness**

Estimation and approximation may be used to check the reasonableness of answers.

Example: Estimate the answer.

$$\frac{58 \times 810}{1989}$$

58 becomes 60, 810 becomes 800 and 1989 becomes 2000.

$$\frac{60 \times 800}{2000} = 24$$

Word problems: An estimate may sometimes be all that is needed to solve a problem.

Example: Janet goes into a store to purchase a CD on sale for $13.95. While shopping, she sees two pairs of shoes, prices $19.95 and $14.50. She only has $50. Can she purchase everything? (Assume there is no sales tax.)

Solve by rounding:

$19.95→$20.00
$14.50→$15.00
$13.95→$14.00
$49.00 Yes, she can purchase the CD and the shoes.

**Apply inductive and deductive reasoning to solve problems.**

**Inductive thinking** is the process of finding a pattern from a group of examples. That pattern is the conclusion that this set of examples seemed to indicate. It may be a correct conclusion or it may be an incorrect conclusion because other examples may not follow the predicted pattern.

**Deductive thinking** is the process of arriving at a conclusion based on other statements that are all known to be true, such as theorems, axiomspostulates, or postulates. Conclusions found by deductive thinking based on true statements will **always** be true.

Examples:

Suppose:

On Monday Mr. Peterson eats breakfast at McDonalds.
On Tuesday Mr. Peterson eats breakfast at McDonalds.
On Wednesday Mr. Peterson eats breakfast at McDonalds.
On Thursday Mr. Peterson eats breakfast at McDonalds again.

Conclusion: On Friday Mr. Peterson will eat breakfast at McDonalds again.

This is a conclusion based on inductive reasoning. Based on several days observations, you conclude that Mr. Peterson will eat at McDonalds. This may or may not be true, but it is a conclusion arrived at by inductive thinking.

A **counterexample** is an exception to a proposed rule or conjecture that disproves the conjecture. For example, the existence of a single non-brown dog disproves the conjecture “all dogs are brown”. Thus, any non-brown dog is a counterexample.

In searching for mathematic counterexamples, one should consider extreme cases near the ends of the domain of an experiment and special cases where an additional property is introduced. Examples of extreme cases are numbers near zero and obtuse triangles that are nearly flat. An example of a special case for a problem involving rectangles is a square because a square is a rectangle with the additional property of symmetry.

Example:

Identify a counterexample for the following conjectures.

1. If $n$ is an even number, then $n$ +1 is divisible by 3.

   $n = 4$
   $n + 1 = 4 + 1 = 5$
   5 is not divisible by 3.

2. If n is divisible by 3, then $n^2 - 1$ is divisible by 4.

   n = 6
   $n^2 - 1 = 6^2 - 1 = 35$
   35 is not divisible by 4.

**Proofs by mathematical induction.**

Proof by induction states that a statement is true for all numbers if the following two statements can be proven:

1. The statement is true for $n = 1$.
2. If the statement is true for $n = k$, then it is also true for $n= k+1$.

In other words, we must show that the statement is true for a particular value and then we can assume it is true for another, larger value ($k$). Then, if we can show that the number after the assumed value ($k$+1) also satisfies the statement, we can assume, by induction, that the statement is true for all numbers.

The four basic components of induction proofs are: (1) the statement to be proved, (2) the beginning step ("let $n$ = 1"), (3) the assumption step ("let $n = k$ and assume the statement is true for $k$, and (4) the induction step ("let $n = k$+1").

Example:

Prove that the sum all numbers from 1 to *n* is equal to $\frac{(n)(n+1)}{2}$.

Let $n = 1$. — Beginning step.

Then the sum of 1 to 1 is 1.

And $\frac{(n)(n+1)}{2} = 1$.

Thus, the statement is true for $n = 1$. — Statement is true in a particular instance.

Assumption:

Let $n = k + 1$

$k = n - 1$

Then $[1 + 2 + \ldots + k] + (k+1) = \frac{(k)(k+1)}{2} + (k+1)$ — Substitute the assumption.

$= \frac{(k)(k+1)}{2} + \frac{2(k+1)}{2}$ — Common denominator.

$= \frac{(k)(k+1)+2(k+1)}{2}$ — Add fractions.

$= \frac{(k+2)(k+1)}{2}$ — Simplify.

$= \frac{(k+1)+1)(k+1)}{2}$ — Write in terms of *k*+1.

For n = 4 , k = 3

$= \frac{(4+1)(4)}{2} = \frac{20}{2} = 10$

Conclude that the original statement is true for $n = k+1$ if it is assumed that the statement is true for $n = k$.

**Proofs on a coordinate plane**

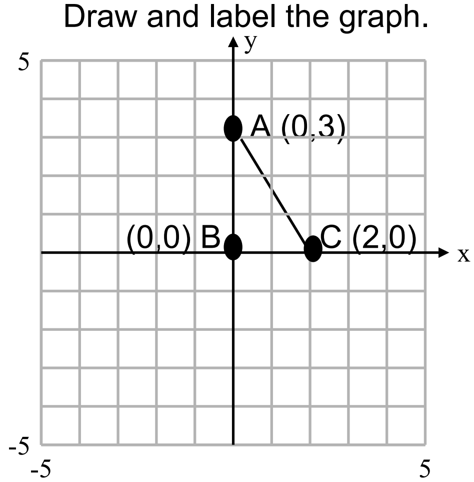
Use proofs on the coordinate plane to prove properties of geometric figures. Coordinate proofs often utilize formulas such as the Distance Formula, Midpoint Formula, and the Slope Formula.

The most important step in coordinate proofs is the placement of the figure on the plane. Place the figure in such a way to make the mathematical calculations as simple as possible.

Example:

1. Prove that the square of the length of the hypotenuse of triangle ABC is equal to the sum of the squares of the lengths of the legs using coordinate geometry.

Draw and label the graph.




Use the distance formula to find the lengths of the sides of the triangle.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

AB = $\sqrt{3^2} = 3$, BC = $\sqrt{2^2} = 2$, AC = $\sqrt{3^2 + 2^2} = \sqrt{13}$

Conclude

$(AB)^2 + (BC)^2 = 3^2 + 2^2 = 13$
$(AC)^2 = (\sqrt{13})^2 = 13$

Thus, $(AB)^2 + (BC)^2 = (AC)^2$